Synchronization in the Kuramoto model

David Gérard-Varet

Université Paris Diderot

with H. Dietert, B. Fernandez, G. Giacomin

1. Background

Motivation: To understand synchronization in a large population of interacting elements: cells of the cardiac pacemaker, flashing of fireflies...

Basic model: N oscillators, described by their

phases $\theta_1, \dots, \theta_N \in \mathbb{T}$, and *natural frequencies* $\omega_1, \dots, \omega_N$.

Free evolution: $\dot{\theta}_i = \omega_i$ (w.l.o.g. $\frac{1}{N}\sum_{i=1}^N \omega_i = 0$).

Mean field coupling: discrete Kuramoto model (1975)

$$\boxed{\dot{\theta}_i = \omega_i - \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_i - \theta_j), \quad i = 1, \dots, N, \quad K > 0} \qquad \text{(KD)}$$

Question: Lim $N \to +\infty$?

Classical mean field theory applies [Golse'13].

Appropriate object is *the empirical measure* :

$$\boxed{f_N(t,\theta,\omega) = \frac{1}{N}\sum \delta_{(\theta_i(t),\omega_i)}(\theta,\omega)}$$

Theorem

Let $f_0 \in \mathcal{P}(\mathbb{T}\times\mathbb{R})$. Assume $f_N(0) \to f_0$.

Then $f_N(t) \to f(t)$ for all $t \geq 0$, f solution in $C(\mathbb{R}_+, w-\mathcal{P}(\mathbb{T}\times\mathbb{R}))$ of

$$\boxed{\partial_t f + \partial_\theta(\omega f - K\int_{\mathbb{T}\times\mathbb{R}} \sin(\theta-\theta')df(\theta',\omega')f) = 0} \qquad \text{(K)}$$

with data f_0.

Remark: Different from Vlasov like equations. Here, transport in θ only.

Remark: $\int f(t, \theta, \omega) d\theta = \int f_0(\theta, \omega) d\theta$.

Interpretation: for ω_i i.i.d. random variables, with law g.

$$\int f_0(\theta, \omega) d\theta = \lim_{N \to \infty} \int f_N(0, \theta, \omega) d\theta = \lim_{N \to \infty} \frac{1}{N} \sum \delta_{\omega_i} = g \quad \text{p.s.}$$

(law of large numbers).

We shall denote: $g(\omega) = \int f(t, \theta, \omega) d\theta$.

2. Qualitative features

Question: Synchronization ? Locking of phases on a common value ?

Related to the so-called *order parameter*:

(KD): $r(t) = \frac{1}{N}\sum_k e^{i\theta_k}$. (K): $\boxed{r(t) = \int_{\mathbb{T}\times\mathbb{R}} e^{i\theta} f(t,\theta,\omega) d\theta d\omega}$.

Asynchrony : $r \approx 0$. Synchronization : $|r| \approx 1$.

Remark: (K) reads

$$\boxed{\partial_t f + \partial_\theta \left(\omega f - \frac{K}{2i}(e^{i\theta}\overline{r(t)} - e^{-i\theta} r(t)) f \right) = 0}$$

Example: $r = 0$. *Incoherent state*: $\boxed{f_i(\theta,\omega) = \frac{g(\omega)}{2\pi}}$

Existence of steady solutions with $r > 0$?

Satisfy $\partial_\theta \left((\omega - Kr\sin\theta) f \right) = 0$.

Two kinds of oscillators, depending on the natural frequency:

- $|\omega| > Kr$: *drifting oscillators*:

$$\boxed{df(\theta,\omega) = \frac{C_\omega}{\omega - Kr\sin\theta} = \frac{\sqrt{\omega^2 - (Kr)^2}}{2\pi|\omega - Kr\sin\theta|} g(\omega) d\omega d\theta}$$

- $|\omega| < Kr$: *locked oscillators*:

  One gets a combination of Dirac masses:

$$\boxed{df(\theta,\omega) = \alpha(\omega) g(\omega) \delta_{\theta_s(\omega)} + (1 - \alpha(\omega)) g(\omega) \delta_{\theta_u(\omega)}}$$

  with

$$\theta_s(\omega) = \arcsin\left(\frac{\omega}{Kr}\right), \quad \theta_u(\omega) = \pi - \arcsin\left(\frac{\omega}{Kr}\right).$$

Remark: if we fix the order parameter in (K) ($r(t) = r$).

Linear equation, with characteristic equation:

$$\dot{\theta} = \omega - Kr\sin(\theta)$$

$\theta_s(\omega)$ stable, $\theta_u(\omega)$ unstable. Suggests stability of

$$\begin{cases} df_s(\theta,\omega) = \dfrac{\sqrt{\omega^2-(Kr)^2}}{2\pi|\omega - Kr\sin\theta|} g(\omega) d\omega d\theta, & |\omega| > Kr \\ df_s(\theta,\omega) = g(\omega)\delta_{\theta_s(\omega)}, & |\omega| < Kr \end{cases}$$

and instability of *possible* other partially locked states.

Warning ! $f_s = f_{s,r}$ must satisfy a self-consistency relation

$$\boxed{r = \int_{\mathbb{T}\times\mathbb{R}} e^{i\theta} f_{s,r}(\theta,\omega) d\theta d\omega}$$

Nonlinear equation in r, with parameter K. Not obvious !

Special case: g *unimodal*, that is

$g : \mathbb{R} \mapsto \mathbb{R}$ continuous, even, decreasing over $\mathbb{R}_+$.

Proposition

- For $K < \frac{2}{\pi g(0)}$, a single solution: f_i, $r = 0$.
- For $K > \frac{2}{\pi g(0)}$, another solution: f_s, $r = r_s > 0$.

Remark: (K) invariant by a shift in θ: circle of synchronized states:
$f_{s,\phi}(\omega, \theta) = f_s(\theta + \phi, \omega)$, with $r_\phi = e^{-i\phi} r$.

Time evolution

Numerics (unimodal case) :

Convergence to f_i for $K < \frac{2}{\pi g(0)}$, $r \to 0$.

Convergence to f_s for $K > \frac{2}{\pi g(0)}$, $r \to r_s$.

Problem: seems impossible in classical functional spaces!

- Classical L^2 norms do not decay.
- Linearized operator f_i or f_s has imaginary spectrum.

Already true for free transport: $\boxed{\partial_t f + \partial_\theta(\omega f) = 0}$.

But … in Fourier space : $(\theta, \omega) \to (l, \xi)$:

$$\boxed{\partial_t \hat{f} - l\partial_\xi \hat{f} = 0, \quad \text{with} \quad \hat{f}(t, l, \xi) = \hat{f}_0(l, \xi + lt)}$$

- Transfer from low to high frequencies : pointwise convergence in Fourier, *i.e.* weak convergence in measures.
- Speed of convergence depends on the smoothness of f_0.

  Example: for analytic f_0, exponential rate.

Conclusion: one can hope for convergence :

- in weak topology
- in strong topology in the moving frame: $\theta' = \theta - \omega t$.

Back to the full equation:

Problems

- Linear analysis: spectral stability of f_i or f_s ?
- Effect of nonlinearities ?

  Difficult:
  - Spectrum is on the imaginary axis in usual spaces.
  - Cascade from high frequencies to low frequencies.

Very limited results: [Crawford'94],[0tt-Antonsen'04,'08],[Carillo et al'13]

But parallel to *Landau damping in plasma physics*...

Nonlinear Landau damping

Refs: [Hwang-Velazquez'09],[Lin-Zheng'11],[Mouhot-Villani'11], [Faou-Rousset'14], [Bedrossian-Masmoudi-Mouhot'16]

About the Vlasov equation !

$$\partial_t f + \partial_x(vf) + \partial_v(Ef) = 0,\ E(x) = -\int_{\mathbb{T}\times\mathbb{R}} \partial_x V(x-x')f(x',v')dx'dv'$$

Typical result

$f_h = f_h(v)$ a homogeneous and smooth equilibrium, spectrally stable.

$f_0 = f_0(x, v)$ smooth, close to f_h.

Then the solution $f = f(t, x, v)$ with data f_0 converges weakly to $\tilde{f}_h = \tilde{f}_h(v)$, homogeneous and smooth equilibrium close to f_h.

Moreover, E strongly converges to 0.

Remarks:

- Spectral stability for hoogeneous states of Vlasov is known: *Penrose condition*. If f_h unimodal, this condition is satisfied.
- E is an integral in v (Fourier : $\xi = 0$). Transfer from low to high frequencies allows for strong convergence.
- Weak convergence of f comes from strong convergence result in the moving frame : $x' = x - vt$.

- Homogeneity and regularity assumptions are crucial, at least in the proofs. VP: theorem is false in H^s, $s < 3/2$.
- Link between the smoothness required and the singularity of kernel V. VP: Gevrey. Vlasov-HMF : Sobolev or C^k.

Consequences on Kuramoto model

Methods of Landau damping can be extended to describe the asynchrony.

Theorem ([Fernandez-Giacomin-GV'14])

Let g s.t. $\|\langle\omega\rangle g\|_{H^4} < +\infty$, $\|\langle|\xi|^4\rangle\hat{g}\|_{L^1} < +\infty$. If

$$(\mathrm{H}) \quad 1 - \frac{K}{2}\int_{\mathbb{R}_+} \hat{g}(\xi) e^{-i\omega\xi} d\omega \neq 0 \quad \forall\omega \text{ with } \mathcal{I}m\,\omega \leq 0$$

there exists ε_K such that: for all f_0 with $||f_0 - f_i||_{H^4} \leq \varepsilon_K$,

$$f(t) \rightharpoonup f_i \ \text{ in } \ L^2, \text{ with } r(t) = O(t^{-4}).$$

Remark: Condition (H) is a spectral stability condition, of Penrose type. If g unimodal, it comes down to $K < \frac{2}{\pi g(0)}$.

Question: What about $K > \frac{2}{\pi g(0)}$? Asymptotic stability of f_s ?

Problem: f_s is inhomogeneous and irregular (Dirac mass). Previous methods do not apply.

3. Study of synchronization

With H. Dietert, B. Fernandez.

Relies on ideas of [Dietert'14].

Main contribution: well-suited functional space, with a "weak" norm.

1. In such space, explicit criterion of spectral stability for the linearization around f_s.

   Completes existing linear results [Mirollo-Strogatz'07].

2. In such space, result of the type:

$$\boxed{\text{Spectral stability} \;\Rightarrow\; \text{Nonlinear asymptotic stability}}$$

New: inhomogeneous and irregular state (Dirac mass).

Functional space

Fourier: $u(l,\xi) = \frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{R}} e^{i(l\theta+\xi\omega)} f(\theta,\omega)d\omega d\theta$.

$$\boxed{\partial_t u(l,\xi) = l\partial_\xi u(l,\xi) + \frac{Kl}{2}\Big(u(1,0)u(l-1,\xi) - \overline{u(1,0)}u(l+1,\xi)\Big)} \tag{KF}$$

Remark: $u(t,0,\xi) = \hat{g}(\xi) \quad \forall t,\ \forall\xi$.

Modes $u(t,l,\xi)$, $l>0$ *and* $l<0$ *are decoupled.*

As $u(l,\xi) = \overline{u(-l,-\xi)}$: we restrict to $l>0$.

Underlying free transport: $\boxed{\partial_t u - l\partial_\xi u = 0, \quad u(t,l,\xi) = u_0(\xi+lt).}$

If $\lim_{+\infty} u_0 = 0$, *then* $\lim_{t\to+\infty} u = 0$.

Different from classical spaces for which $\lim_{\pm\infty} u_0 = 0$.

Concretely, for $a > 0$:

$$\mathcal{P}_a = \Big\{ f \in \mathcal{P}_1(\mathbb{T} \times \mathbb{R}), \ \text{ such that } \ \hat{f}(0,\xi) = \hat{g}(\xi) \text{ et } \sum_{l \in \mathbb{N}} \int_{\mathbb{R}} e^{2a\xi} (|\hat{f}(l,\xi)|^2 + |\partial_\xi \hat{f}(l,\xi)|^2) d\xi < \infty \Big\},$$

Distance over $\mathcal{P}_a$:

$$d_a(f,g) = \sqrt{\sum_{l \in \mathbb{N}} \int_{\mathbb{R}} e^{2a\xi} (|(\hat{f} - \hat{g})(l,\xi)|^2 + |\partial_\xi (\hat{f} - \hat{g})(l,\xi)|^2) d\xi}$$

Important: $e^{2a\xi} \neq e^{2a|\xi|}$: the latter corresponds to analytic setting (Paley-Wiener).

Remark: Use of exponential weight not new : see [Pego-Weinstein'94] (stability of the KDV soliton).

- Well-suited to partially locked states f_s:

Proposition

$$\boxed{f_s \in \mathcal{P}_a, \quad f_u \notin \mathcal{P}_a}$$

Remark: Important to exclude f_u.

- $\mathcal{P}_a$ is preserved by the Kuramoto flow

Proposition

If $\displaystyle\int_{\mathbb{R}} e^{2a|\xi|} \left(|\hat{g}(\xi)|^2 + |\partial_\xi \hat{g}(\xi)|^2\right) d\xi < +\infty$,
and if $f_0 \in \mathcal{P}_a$, then the solution of (K) satisfies $f \in C(\mathbb{R}_+, \mathcal{P}_a)$.

Linear stability

Linearization around f_s in Fourier: $u := \hat{f} - \hat{f}_s$, $u_s := \hat{f}_s$. One gets

$$\boxed{\partial_t u = Lu = L_1 u + L_2 u}$$

with, for $l \in \mathbb{N}_*$, $\xi \in \mathbb{R}$:

$$\boxed{\begin{aligned} L_1 u(l,\xi) &:= l\partial_\xi u(l,\xi) + \frac{Kl}{2} r_s \left(u(l-1,\xi) - u(l+1,\xi)\right) \\ L_2 u(l,\xi) &:= \frac{Kl}{2}\left(u(1,0)u_s(l-1,\xi) - \overline{u(1,0)}u_s(l+1,\xi)\right). \end{aligned}}$$

Technical difficulty: L_2 $\mathbb{R}$-linear, not $\mathbb{C}$-linear.

But $\mathbb{C}$-linear in $(u, \overline{u})$. Allows for complexification of the operator. The spectrum mentioned below is the one of the complexified operator.

Operators are considered over

$$\mathcal{Z}_a = \left\{ u,\ u(0,\xi) = 0,\ \sum_{l \in \mathbb{N}} \int_{\mathbb{R}} e^{2a\xi} (|\hat{f}(l,\xi)|^2 + |\partial_\xi \hat{f}(l,\xi)|^2) d\xi < \infty \right\},$$

Theorem

1. $\sigma_{ess}(L) = \sigma_{ess}(L_1) \subset \{\mathcal{R}e\,\lambda \leq -a\}$.
2. $\forall \eta > 0, \quad vp(L) \cap \{\mathcal{R}e\,\lambda \geq -a + \eta\}$ is finite, with explicit characterization.
3. $0 \in vp(L)$

Remark: the first item is crucial: $\sigma_{ess} \not\subset i\mathbb{R}$.

Remark: The third item comes from the shift invariance of (K) in θ.

Proof of the first item:

- L_2 of finite rank, so that $\sigma_{ess}(L) = \sigma_{ess}(L_1)$.

- Property $\sigma_{ess}(L_1) \subset \{\mathcal{R}e\, \lambda \leq -a\}$: *a priori* estimate for the resolvent equation:

$$\boxed{\lambda u(l,\xi) - l\partial_\xi u(l,\xi) - \frac{Kl}{2} r_s \left(u(l-1,\xi) - u(l+1,\xi)\right) = f(l,\xi)}$$

Testing against $l^{-1} e^{2a\xi} \overline{u(l,\xi)}$, one finds

$$\frac{1}{2}(\mathcal{R}e\, \lambda)\|l^{-1}u\|^2_{\mathcal{Z}_a} \;+\; a\|u\|^2_{\mathcal{Z}_a} \;\leq\; \|l^{-1}f\|_{\mathcal{Z}_a}\, \|u\|_{Z_a}$$

Nonlinear stability

Spectral stability assumption ($a > 0$ fixed):

$$\text{(H)} \quad \sigma(\mathcal{L}) \cap \{\mathcal{R}e\, \lambda \geq 0\} = \{0\}$$

Proposition ([Mirollo-Strogatz'07]

If g is unimodal, (H) is satisfied for all $K > \frac{2}{\pi g(0)}$.

Theorem

Under assumption (H): there exists $\varepsilon > 0$, $a' > 0$ s.t. for all data $f_0 \in \mathcal{P}_a$ with $d_a(f_0, f_s) \leq \varepsilon$, one can find $\phi = \phi(f_0)$ satisfying:

$$d_a(f(t), f_s(\cdot + \phi, \cdot)) \leq C_\varepsilon\, e^{-a' t}$$

Idea of proof:

Center manifold theorem.

Here, the center manifold is explicit: circle of steady stable solutions :

$$\mathcal{V} = \{e^{i\varphi} u_s - u_s, \quad \varphi \in \mathbb{R}\}$$

Technical difficulty : the nonlinearity does not preserve $\mathcal{Z}_a$.

Use of a regularizing effect of the semigroup L^2 in time....