Long time behavior of solutions to the 2D Keller-Segel equation

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The Keller-Segel equation

- The Keller-Segel equation models the collective motion of cells attracted by a self-emitted chemical substance. The parabolic-elliptic Keller-Segel equation in 2D is

$$\rho_t = \Delta\rho + \nabla \cdot (\rho \nabla(\mathcal{N} * \rho)),$$

where $\mathcal{N} = \frac{1}{2\pi} \log |x|$ is the Newtonian potential in $\mathbb{R}^2$.

(Patlak '53, Keller-Segel '71)

- There exists a "critical mass" $M_c = 8\pi$ such that:
  - If the mass satisfies $M < M_c$, the solution remains globally bounded; if $M > M_c$, solutions blow-up in finite time.

  (Jager-Luckhaus '92, Nagai '01, Dolbeault-Perthame '04, Bedrossian-Masmoudi '14)

  - If $M = M_c$, the global solution may aggregate in infinite time.

  (Blanchet-Carrillo-Masmoudi '08, Blanchet-Carlen-Carrillo '12, Carlen-Figalli '13)

Keller-Segel equation with nonlinear diffusion

- In $\mathbb{R}^d$ with $d \geq 2$, the K-S equation with nonlinear diffusion is

$$\rho_t = \Delta \rho^m + \nabla \cdot (\rho \nabla (\mathcal{N} * \rho)),$$

  where $\mathcal{N}$ is the Newtonian potential in $\mathbb{R}^d$. Here we assume $m > 1$, which models the anti-overcrowding effect. (Boi-Capasso-Morale '00, Topaz-Bertozzi-Lewis '06)

- The behavior of solutions depends on m, with $m_c = 2 - \frac{2}{d}$ being the critical power:
  - For $m > m_c$, for any $\rho_0 \in L^1 \cap L^\infty(\mathbb{R}^d)$, the solution exists globally in time, and the L^∞ norm stays uniformly bounded in time. (Sugiyama '06)
  - For $m < m_c$, there might be a finite-time blow-up for initial data with arbitrarily small mass. (Sugiyama '06)
  - For $m = m_c$, the behavior of solution depends on its mass, and there is a critical mass M_c. (Blanchet-Carrillo-Laurençot '09)

- From now on, we focus on the "subcritical" case $m > 2 - \frac{2}{d}$, where the solutions are known to exist globally in time.
- Question: long time behavior of solutions?
- If ρ is a solution the Keller-Segel equation, then $E_m[\rho]$ is non-increasing in time:

$$E_m[\rho] = \frac{1}{m-1} \int_{\mathbb{R}^d} \rho^m dx + \frac{1}{2} \int_{\mathbb{R}^d} \rho(\mathcal{N} * \rho) dx.$$

- Let ρ_A be the global minimizer of E_m among all densities with mass A. Then ρ_A must be a stationary solution.
- The following are known about the global minimizer ρ_A:
  - Existence (Lions '84)
  - Radial symmetry (by Riesz's rearrangement inequality)
  - Uniqueness + compact support (Lieb-Yau '87)

Question

If ρ_0 has mass A, is it always true that $\rho(\cdot, t)$ converges to (a translation of) ρ_A as $t \to \infty$?

- The answer is yes ONLY IF we have a positive answer to the following question:

Question

Is ρ_A the unique stationary solution with mass A (up to a translation)?

- For Newtonian potential, it is known that radially symmetric stationary solution (with a fixed mass) is unique (Lieb-Yau '87), so the above question is equivalent with

Question

Is every stationary solution radially symmetric (up to a translation)?

Stationary solutions for aggregation equation with nonlinear diffusion

Consider the equation with a general attracting kernel $\mathcal{K}$:

$$\rho_t = \Delta \rho^m + \nabla \cdot \big(\rho \nabla (\mathcal{K} * \rho)\big),$$

where $\mathcal{K}$ is radial and is strictly increasing in $|x|$. Thus any stationary solution ρ_s satisfies

$$\frac{m}{m-1}\rho_s^{m-1} + \mathcal{K} * \rho_s = C_i$$

in each connected component of $\{\rho_s > 0\}$. (C_i can differ in different components).

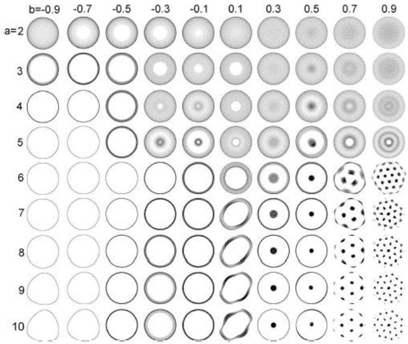
Theorem (Carrillo-Hittmeir-Volzone-Y., '16)

Let $\rho_s \in L^1_+(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ be a stationary solution in the above sense. Then ρ_s must be radially decreasing up to a translation.

Contrast with the attractive-repulsive kernel

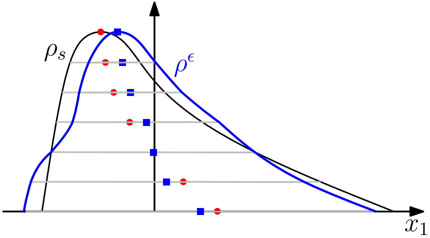
If K is repulsive in short-range and attracting in long-range, then stationary solutions to $\rho_t = \nabla \cdot \big(\rho \nabla (\mathcal{K} * \rho)\big)$ can have many non-radial patterns.

For example, when $\mathcal{K}'(r) = \tanh((1-r)a) + b$ with parameters a, b, below are the patterns of stationary solutions for some a, b: (Kolokolnikov-Sun-Uminsky-Bertozzi, '11)



Sketch of the proof

- If a stationary solution ρ_s that is NOT radially decreasing after any translation, we perturb it using its continuous Steiner symmetrization about some hyperplane H:



- Since $\int \rho_s^m = \int (\rho^\epsilon)^m$, and interaction energy decreases in the first order for a short time (need some work to check this!),

$$E_m[\rho^\epsilon] - E_m[\rho_s] < -c\epsilon \quad \text{for all sufficiently small } \epsilon > 0,$$

where $c > 0$ depending on ρ_s and $\mathcal{K}$.

Working towards a contradiction

- $E_m[\rho^\epsilon] - E_m[\rho_s] < -c\epsilon$ does not directly lead to a contradiction: ρ_s is only Hölder continuous near the zero levelset, hence we may have $\|\rho^\epsilon - \rho_s\|_\infty \sim \epsilon^{1/(m-1)}$.
- Let us now slow down the "velocity" at low density $h < h_0$ as $v(h) = (h/h_0)^{m-1}$, and call the perturbation μ^ϵ. We still have

$$E_m[\mu^\epsilon] - E_m[\rho_s] < -c\epsilon \quad \text{for all sufficiently small } \epsilon > 0,$$

  and using a priori regularity estimates on ρ_s gives

$$|\mu^\epsilon(x) - \rho_s(x)| \leq C\epsilon|\rho_s(x)| \quad \text{for all sufficiently small } \epsilon > 0.$$

- Combining the above pointwise estimate with the assumption that ρ_s is stationary, we have $|E_m[\mu^\epsilon] - E_m[\rho_s]| < C\epsilon^2$, contradicting the first inequality if $\epsilon > 0$ is sufficiently small. So there cannot be such a ρ_s!

Convergence of dynamical solution in 2D

- For any $t_n \to \infty$, weak lower semicontinuity of the entropy dissipation (Bian-Liu '13) gives that $\|\rho(\cdot, t_{n_k}) - \rho_\infty\|_{L^1} \to 0$ for some stationary solution ρ_∞ along a subsequence $t_{n_k} \to \infty$.
- ρ_∞ has the same center of mass as ρ_0: center of mass is preserved during evolution.
- ρ_∞ also has the same mass as ρ_0: the second moment $\int |x|^2 \rho(x,t)dx$ is uniformly bounded in time. (This argument works in 2D only!)

Theorem (Carrillo-Hittmeir-Volzone-Y., '16)

For any $\rho_0 \in L^\infty(\mathbb{R}^2) \cap L^1((1+|x|^2)dx)$, we have

$$\lim_{t\to\infty} \|\rho(\cdot,t) - \rho_s\|_{L^q} = 0 \text{ for any } 1 \le q < \infty,$$

where ρ_s is the (unique) stationary solution with the same mass and same center of mass as ρ_0.

- But we are unable to obtain any convergence rates.

The "$m = \infty$" limit (joint with K.Craig and I.Kim)

If we take the "$m \to \infty$" limit in $\rho_t = \Delta\rho^m + \nabla \cdot (\rho\nabla(\rho * \mathcal{N}))$, $\rho(\cdot, t)$ should intuitively evolve like congested penguins:

Gradient flow with density constraint

- Recall: For Keller-Segel equation with power m, its associated free energy functional is

$$E_m[\rho] = \frac{1}{m-1}\int_{\mathbb{R}^d} \rho^m dx + \frac{1}{2}\int_{\mathbb{R}^d} \rho(\mathcal{N} * \rho) dx.$$

One can easily check that $\lim_{m\to\infty} E_m[\rho] = E_\infty[\rho]$ for any ρ, where

$$E_\infty[\rho] := \begin{cases} \frac{1}{2}\int_{\mathbb{R}^d} \rho(\mathcal{N} * \rho) dx & \text{if } \|\rho\|_\infty \le 1, \\ +\infty & \text{otherwise.} \end{cases}$$

- Our goal is to study the properties of gradient flow of E_∞.
- Such problem has been studied when $\mathcal{N} * \rho$ is replaced by a fixed potential Φ. (Maury-RoudneffChupin-Santambrogio '10, Alexander-Kim-Yao '14)

Let

$$E_\infty[\rho] := \begin{cases} \frac{1}{2}\int_{\mathbb{R}^d} \rho(\mathcal{N} * \rho)dx & \text{if } \|\rho\|_\infty \le 1, \\ +\infty & \text{otherwise.} \end{cases}$$

Question

1. Given $\rho_0 \in \mathcal{P}_2(\mathbb{R}^d)$ with $\|\rho_0\|_\infty \le 1$, is the gradient flow of E_∞ well defined, and is it unique?
2. If $\rho_\infty(\cdot, t)$ is the gradient flow of E_∞, what PDE does it satisfy?
3. Long time behavior of $\rho_\infty(\cdot, t)$?

- Question 1 has a positive answer: Thanks to the L^∞ constraint, the energy E_∞ has certain convexity properties along generalized geodesics (Carrillo-Lisini-Mainini '14), hence the continuous gradient flow to E_∞ is well defined (Craig '15).

Evolution of solutions with "patch type" initial data

- Let us consider the initial data $\rho_0 \in \mathcal{P}_2(\mathbb{R}^d)$ be of "patch type", that is, $\rho_0 = 1_{\Omega_0}$.
- Without the density constraint, a solution for the aggregation equation $\rho_t = \nabla \cdot (\rho \nabla(\rho * \mathcal{N}))$ remains a patch during its existence, whose density blows up in finite time. (Bertozzi-Laurent-Leger '12)
- For the gradient flow of E_∞, if ρ_0 is of patch type, intuitively we expect that $\rho(t)$ stays a patch $\rho(t) = 1_{\Omega(t)}$, due to the attracting Newtonian potential.

Question

What PDE determines the evolution of $\Omega(t)$?

PDE for patch solutions

- A heuristic argument suggests that if $\rho(t) = 1_{\Omega(t)}$, then $\rho(t)$ should satisfy the transport equation

$$\rho_t = \nabla \cdot (\rho(\nabla(\rho * \mathcal{N}) + \nabla u)),$$

where the "corrector" u satisfies

$$\begin{cases} \Delta u = -1 \text{ in } \Omega(t), \\ u = 0 \text{ on } \partial\Omega(t). \end{cases}$$

- So the free boundary $\partial\Omega(t)$ should evolve like a Hele-Shaw type equation, with free boundary velocity given by

$$V(x,t) = -\nabla(\mathcal{N} * 1_{\Omega(t)}) \cdot \vec{n} + |\nabla u|$$

Theorem (Craig-Kim-Y., '16)

Let $\rho_\infty(\cdot, t)$ be the gradient flow of E_∞ with $\rho_0 = 1_{\Omega_0}$. Then we have $\rho_\infty(t) = 1_{\Omega(t)}$ a.e. for all time, where $\Omega(t)$ is a viscosity solution of the above free boundary problem.

Convergence towards a disk

Question

Long-time behavior of patch solutions?

Theorem (Craig-Kim-Y., '16)

For $d = 2$ and $\rho_0 = 1_{\Omega_0}$, we have $\lim_{t\to\infty} \|\rho(t) - 1_B\|_{L^q} = 0$ for any $1 \le q < \infty$, where B is a disk with area 1 whose center coincides with the center of mass of ρ_0. Moreover, we have

$$0 \le E_\infty[\rho(t)] - E_\infty[1_B] \le C(M_2[\rho_0])t^{-1/6} \text{ for all } t \ge 0.$$

- Reason: the evolution of second moment (in 2D) is given by

$$\frac{d}{dt}M_2[\rho(t)] = -\frac{1}{2\pi} + 4\int_{\Omega(t)} u(x)dx,$$

  where $\Delta u = -1$ in $\Omega(t)$ and $u = 0$ on $\partial\Omega(t)$.
- Using similar ideas as Talenti '76, we have RHS ≤ 0, where the equality is achieved if and only if Ω is a disk.

Idea of convergence proof

- Using a stability result of isoperimetric inequality (Fusco-Maggi-Pratelli '08), we have

$$-\frac{1}{2\pi} + 4\int_{\Omega(t)} u(x)dx \lesssim -A(\Omega(t))^3,$$

where

$$A(E) := \inf\left\{\frac{|E\triangle(x_0+B)|}{|E|} : x_0 \in \mathbb{R}^2, B \text{ is a disk with } |B| = |E|\right\}.$$

- This result enables us to get some (non-optimal) convergence rate for $d = 2$.
- For $d \geq 3$, we are unable to obtain any compactness result of $\rho(t)$ as $t \to \infty$, therefore it is unknown whether $\Omega(t)$ converges to a ball as $t \to \infty$.

Thank you for your attention!