Fluctuations and deviations in MFG

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Joint works: Cardaliaguet, Lasry and Lions; Carmona; Lacker and Ramanan

Part I. Motivation

N-Particle System

- N interacting controlled players (state in $\mathbb{R}^d$)
  - dynamics of player number $i \in \{1, \dots, N\}$

$$dX_t^i = \alpha_t^i dt + dW_t^i \qquad , \quad X_0^i = x_0, \; t \in [0, T]$$

  - independent noises $W^1, \dots, W^N$,
  - choose control $\underbrace{\alpha_t^i}_{\text{at any } t}$ = prog. meas. w.r.t. $\sigma(W^1, \dots, W^N, \quad)$

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- N interacting controlled players (state in $\mathbb{R}^d$)
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$$dX_t^i = \alpha_t^i dt + dW_t^i + \sqrt{\eta} dB_t, \quad X_0^i = x_0, \ t \in [0, T]$$

  - independent noises $W^1, \dots, W^N, B, \ \eta > 0$
  - choose control $\underbrace{\alpha_t^i}_{\text{at any } t} = \text{prog. meas. w.r.t. } \sigma(W^1, \dots, W^N, B)$

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  - independent noises $W^1, \dots, W^N$, B, $\bar{\mu}_t^N = \frac{1}{N} \sum_{j=1}^{N} \delta_{X_t^j}$
  - choose control $\alpha_t^i = \alpha^i(t, X_t^1, \cdots, X_t^N) \rightsquigarrow$ implicit formulation

N-Particle System

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$$dX_t^i = \alpha_t^i dt + dW_t^i + \sqrt{\eta} dB_t, \quad X_0^i = x_0, \; t \in [0, T]$$

  - independent noises $W^1, \dots, W^N$, B, $\bar{\mu}_t^N = \frac{1}{N} \sum_{j=1}^N \delta_{X_t^j}$
  - choose control $\alpha_t^i = \alpha^i(t, X_t^1, \cdots, X_t^N) \rightsquigarrow$ implicit formulation
- Willing to minimize cost $J^i(\alpha^1, \dots, \alpha^N)$ with mean-field interaction

$$J^i(\dots) = \mathbb{E}\Big[g(X_T^i, \bar{\mu}_T^N) + \int_0^T f(X_t^i, \bar{\mu}_t^N, \alpha_t^i) dt\Big]$$

  - $g(x, \mu)$ and $f(x, \mu, \alpha)$ with $x \in \mathbb{R}^d$, $\mu \in \mathcal{P}(\mathbb{R}^d)$ and $\alpha \in A \subset \mathbb{R}^k$
  - f convex in $\alpha \rightsquigarrow$ typical instance $f(x, \mu, \alpha) = f(x, \mu) + \frac{1}{2}|\alpha|^2$

Nash equilibrium

• If each particle / player decides in its own way to minimize

$$J^i(\alpha^1, \dots, \alpha^N)$$

◦ depends on the others! $\Rightarrow$ consensus? $\rightsquigarrow$ Nash equilibrium

• N-tuple $(\alpha^{1,\star}, \dots, \alpha^{N,\star})$ = equilibrium if no incentive to quit

◦ if unilateral change $\alpha^{i,\star} \rightsquigarrow \alpha^i \Rightarrow J^i \nearrow$

$$J^i(\alpha^{1,\star}, \dots, \alpha^{i,\star}, \dots, \alpha^{N,\star}) \leq J^i(\alpha^{1,\star}, \dots, \alpha^i, \dots \alpha^{N,\star})$$

• Meaning of the freezing $\alpha^{1,\star}, \dots, \alpha^{i-1,\star}, \alpha^{i+1,\star}, \alpha^{N,\star}$?

◦ closed loop control $\rightsquigarrow$ $\alpha^i_t = \alpha^i(t, X^1_t, \dots, X^N_t)$ $\rightsquigarrow$ players choose their strategy depending on the states of the others $\rightsquigarrow$ SDE

◦ freezing means freezing the functions $\alpha^{\star,1}, \dots, \alpha^{\star,N}$ and not the processes

• N-particle system $\rightsquigarrow$ N-player game

Nash System

- N fixed $\rightsquigarrow$ N player game equilibrium described by PDE system
  - unique Markovian equilibrium with bounded feedback function

$\rightsquigarrow$ given by $N \times (Nd)$ Nash system $\rightsquigarrow$ $v^{N,i}$ value function to player i

$$\begin{aligned}\partial_t v^{N,i}(t,\boldsymbol{x}) + \frac{1}{2}\sum_j \Delta_{x_j} v^{N,i}(t,\boldsymbol{x}) + \frac{\eta}{2}\sum_{j,k} \mathrm{Tr}\partial^2_{x_j,x_k} v^{N,i}(t,\boldsymbol{x}) \\ - \sum_{j\neq i} \partial_{x_j} v^{N,j}(t,\boldsymbol{x}) \cdot \partial_{x_j} v^{N,i}(t,\boldsymbol{x}) \\ - \frac{1}{2}|\partial_{x_i} v^{N,i}(t,\boldsymbol{x})|^2 + f(x_i, \bar{\mu}^N_{\boldsymbol{x}}) = 0\end{aligned}$$

  - mean field $\bar{\mu}^N_{\boldsymbol{x}} = \frac{1}{N}\sum_{j=1}^N \delta_{x_j}$ $\quad \boldsymbol{x} = (x_1, \cdots, x_N) \in (\mathbb{R}^d)^N$
  - boundary condition $v^{N,i}(T,\boldsymbol{x}) = g(x_i, \bar{\mu}^N_{\boldsymbol{x}})$

- $v^{N,i}(t,\boldsymbol{x})$ = equilibrium cost to player i when

the system starts from $\boldsymbol{x}$ at time t

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$$
\begin{aligned}
\partial_t v^{N,i}(t,\boldsymbol{x}) + \frac{1}{2}\sum_j \Delta_{x_j} v^{N,i}(t,\boldsymbol{x}) + \frac{\eta}{2}\sum_{j,k} \mathrm{Tr}\partial^2_{x_j,x_k} v^{N,i}(t,\boldsymbol{x}) \\
- \sum_{j\neq i} \partial_{x_j} v^{N,j}(t,\boldsymbol{x}) \cdot \partial_{x_j} v^{N,i}(t,\boldsymbol{x}) \\
- \frac{1}{2}|\partial_{x_i} v^{N,i}(t,\boldsymbol{x})|^2 + f(x_i, \bar{\mu}^N_{\boldsymbol{x}}) = 0
\end{aligned}
$$

  - mean field $\bar{\mu}^N_{\boldsymbol{x}} = \frac{1}{N}\sum_{j=1}^N \delta_{x_j}$ $\quad \boldsymbol{x} = (x_1, \cdots, x_N) \in (\mathbb{R}^d)^N$
  - boundary condition $v^{N,i}(T,\boldsymbol{x}) = g(x_i, \bar{\mu}^N_{\boldsymbol{x}})$

- Trajectories at equilibrium

$$
dX^i_t = -\partial_{x_i} v^{N,i}(t, X^1_t, \cdots, X^N_t)dt + dW_t + \sqrt{\eta}dB_t
$$

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$$\partial_t v^{N,i}(t,\boldsymbol{x}) + \frac{1}{2}\sum_j \Delta_{x_j} v^{N,i}(t,\boldsymbol{x}) + \frac{\eta}{2}\sum_{j,k} \mathrm{Tr}\,\partial^2_{x_j,x_k} v^{N,i}(t,\boldsymbol{x})$$

$$- \sum_{j\neq i} \partial_{x_j} v^{N,j}(t,\boldsymbol{x}) \cdot \partial_{x_j} v^{N,i}(t,\boldsymbol{x})$$

$$- \frac{1}{2}|\partial_{x_i} v^{N,i}(t,\boldsymbol{x})|^2 + f(x_i, \bar{\mu}^N_{\boldsymbol{x}}) = 0$$

- mean field $\bar{\mu}^N_{\boldsymbol{x}} = \frac{1}{N}\sum_{j=1}^N \delta_{x_j}$ $\quad \boldsymbol{x} = (x_1, \cdots, x_N) \in (\mathbb{R}^d)^N$
- boundary condition $v^{N,i}(T,\boldsymbol{x}) = g(x_i, \bar{\mu}^N_{\boldsymbol{x}})$

- Well-posed system with bounded gradient and solution is symmetric

$$v^{N,i}(t,\boldsymbol{x}) = v^N(t, x_i, (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots))$$

$v^N(\cdot,\cdot)$ symmetric in the second argument

Nash System

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$$\partial_t v^{N,i}(t,\boldsymbol{x}) + \frac{1}{2}\sum_j \Delta_{x_j} v^{N,i}(t,\boldsymbol{x}) + \frac{\eta}{2}\sum_{j,k} \mathrm{Tr}\partial^2_{x_j,x_k} v^{N,i}(t,\boldsymbol{x})$$
$$- \sum_{j\neq i} \partial_{x_j} v^{N,j}(t,\boldsymbol{x}) \cdot \partial_{x_j} v^{N,i}(t,\boldsymbol{x})$$
$$- \frac{1}{2}|\partial_{x_i} v^{N,i}(t,\boldsymbol{x})|^2 + f(x_i, \bar{\mu}^N_{\boldsymbol{x}}) = 0$$

  - mean field $\bar{\mu}^N_{\boldsymbol{x}} = \frac{1}{N}\sum_{j=1}^N \delta_{x_j}$ $\quad \boldsymbol{x} = (x_1, \cdots, x_N) \in (\mathbb{R}^d)^N$
  - boundary condition $v^{N,i}(T,\boldsymbol{x}) = g(x_i, \bar{\mu}^N_{\boldsymbol{x}})$

- Guess is $v^{N,i}(t,\boldsymbol{x}) \approx \mathcal{U}(t, x_i, \bar{\mu}^N_{\boldsymbol{x}})$ with $\mathcal{U} : [0,\mathrm{T}] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$
  - $\frac{1}{N}\sum_{i=1}^N \delta_{X^i_t}$ should be close the empirical distribution of

$$d\hat{X}^i_t = -\partial_x \mathcal{U}(t, \hat{X}^i_t, \frac{1}{N}\sum_{i=1}^N \delta_{\hat{X}^i_t})dt + dW^i_t + \sqrt{\eta}dB_t$$

Part II. Master equation

Differential calculus on Wasserstein space

- Goal is to write a PDE for $\mathcal{U}$ by plugging $u^{N,i}(t,\boldsymbol{x}) = \mathcal{U}(t, x_i, \bar{\mu}^N_{\boldsymbol{x}})$ as nearly solution of Nash
  - use differential calculus on $\mathcal{P}_2(\mathbb{R}^d) \rightsquigarrow$ Lions' approach
- Given $\mathcal{U} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \rightsquigarrow$ define lifting of $\mathcal{U}$

$$\hat{\mathcal{U}} : L^2(\Omega, \mathbb{P}) \ni X \mapsto \mathcal{U}(\mathcal{L}(X) = \mathrm{Law}(X))$$

  - $\mathcal{U}$ differentiable if $\hat{\mathcal{U}}$ Fréchet differentiable
- Differential of $\mathcal{U} \rightsquigarrow$ Fréchet derivative of $\hat{\mathcal{U}}$

$$D\hat{\mathcal{U}}(X) = \partial_\mu \mathcal{U}(\mu)(X), \quad \partial_\mu \mathcal{U}(\mu) : \mathbb{R}^d \ni x \mapsto \partial_\mu \mathcal{U}(\mu)(x) \quad \mu = \mathcal{L}(X)$$

  - derivative of $\mathcal{U}$ in $\mu \rightsquigarrow \partial_\mu \mathcal{U}(\mu) \in L^2(\mathbb{R}^d, \mu; \mathbb{R}^d)$
- Finite-dimensional projection

$$\partial_{x_i}\Big[\mathcal{U}\Big(\frac{1}{N}\sum_{j=1}^N \delta_{x_j}\Big)\Big] = \frac{1}{N}\partial_\mu \mathcal{U}\Big(\frac{1}{N}\sum_{j=1}^N \delta_{x_j}\Big)(x_i), \quad x_1, \ldots, x_N \in \mathbb{R}^d$$

- Example: $U(\mu) = \int_{\mathbb{R}^d} h(y) d\mu(y) \Rightarrow \partial_\mu U(\mu)(v) = \nabla h(v)$

Second-order differentiability

- Need for existence of second-order derivatives
  - asking the lift to be twice Fréchet is too strong
  - only discuss the existence of second-order partial derivatives
- Requires
  - $\partial_\mu \mathcal{U}(\mu)(v)$ is differentiable in v and μ

$$\partial_v \partial_\mu \mathcal{U}(\mu)(v) \qquad \partial^2_\mu \mathcal{U}(\mu)(v, v')$$

  - $\partial_v \partial_\mu \mathcal{U}(\mu)(v)$ and $\partial^2_\mu \mathcal{U}(\mu)(v, v')$ continuous in (μ, v, v') (for W_2 in μ) with suitable growth
- Finite-dimensional projection

$$\partial^2_{x_i\, x_j}\left[\mathcal{U}\left(\frac{1}{N}\sum_{k=1}^N \delta_{x_k}\right)\right] = \frac{1}{N}\partial_v \partial_\mu \mathcal{U}\left(\frac{1}{N}\sum_{k=1}^N \delta_{x_k}\right)(x_i)\, \delta_{i,j} + \frac{1}{N^2}\partial^2_\mu \mathcal{U}\left(\frac{1}{N}\sum_{k=1}^N \delta_{x_k}\right)(x_i, x_j)$$

Connection with the master equation

- Strategy is to regard $u^{N,i}(t,\boldsymbol{x}) = \mathcal{U}(t, x_i, \bar{\mu}^N_{\boldsymbol{x}})$ as nearly solution
- First-order terms

$$\partial_{x_j} u^{N,i}(t,\boldsymbol{x}) = \begin{matrix} \partial_x \mathcal{U}(t, x_i, \bar{\mu}^N_{\boldsymbol{x}}) + O(\frac{1}{N}) & \text{if} \quad j = i \\ \frac{1}{N}\partial_\mu \mathcal{U}(t, x_i, \bar{\mu}^N_{\boldsymbol{x}})(x_j) & \text{if} \quad j \neq i \end{matrix}$$

  - Hamiltonian

$$-\frac{1}{2}|\partial_{x_i} u^{N,i}(t,\boldsymbol{x})|^2 + f(x_i, \bar{\mu}^N_{\boldsymbol{x}}) = -\frac{1}{2}|\partial_x \mathcal{U}(t, x_i, \bar{\mu}^N_{\boldsymbol{x}})|^2 + f(x_i, \bar{\mu}^N_{\boldsymbol{x}}) + O(\frac{1}{N})$$

  - drift terms

$$\begin{aligned} &-\sum_{j \neq i} \partial_{x_j} u^{N,j}(t,\boldsymbol{x}) \cdot \partial_{x_j} u^{N,i}(t,\boldsymbol{x}) \\ &\quad = -\frac{1}{N}\sum_{j \neq i} \partial_x \mathcal{U}(t, x_j, \bar{\mu}^N_{\boldsymbol{x}}) \cdot \partial_\mu \mathcal{U}(t, x_i, \bar{\mu}^N_{\boldsymbol{x}})(x_j) + O(\frac{1}{N}) \\ &\quad = -\int_{\mathbb{R}^d} \partial_x \mathcal{U}(t, v, \bar{\mu}^N_{\boldsymbol{x}}) \cdot \partial_\mu \mathcal{U}(t, x_i, \bar{\mu}^N_{\boldsymbol{x}})(v) d\bar{\mu}^N_{\boldsymbol{x}}(v) + O(\frac{1}{N}) \end{aligned}$$

  - up to $O(\frac{1}{N}) \rightsquigarrow$ yields first order terms of a PDE for $\mathcal{U}$

Form of the master equation

- Treat second order terms in the same way and get that $\mathcal{U}$ should satisfy Master equation at order 2

$$
\begin{aligned}
&\partial_t \mathcal{U}(t,x,\mu) - \int_{\mathbb{R}^d} \partial_x \mathcal{U}(t,v,\mu) \cdot \partial_\mu \mathcal{U}(t,x,\mu,v) d\mu(v) \\
&\quad - \frac{1}{2}|\partial_x \mathcal{U}(t,x,\mu)|^2 + f(x,\mu) + \frac{1}{2}(1+\eta)\mathrm{Trace}\Big(\partial_x^2 \mathcal{U}(t,x,\mu)\Big) \\
&\quad + \frac{1}{2}(1+\eta) \int_{\mathbb{R}^d} \mathrm{Trace}\Big(\partial_v \partial_\mu \mathcal{U}(t,x,\mu)(v)\Big) d\mu(v) \\
&\quad + \eta \int_{\mathbb{R}^d} \mathrm{Trace}\Big(\partial_x \partial_\mu \mathcal{U}(t,x,\mu)(v)\Big) d\mu(v) \\
&\quad + \frac{1}{2}\eta \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathrm{Trace}\Big(\partial_\mu^2 \mathcal{U}(t,x,\mu)(v,v')\Big) d\mu(v) d\mu(v') = 0
\end{aligned}
$$

- Not a proof of existence of a smooth solution!
  - This should be proved first

Connection with the MFG system $(\eta = 0)$

- Regard $\mathcal{U}$ as the generalized value function of the MFG system
  - $\mathcal{U}(t_0, x_0, \mu^0) = u^{\mu:\mu_{t_0}=\mu^0}(t_0, x_0)$
- Optimization in environment $(\mu_t)_{t\in[0,T]} \rightsquigarrow$ HJB equation
  - $u(t,x)$ = minimal cost under $(\mu_t)_{t\in[0,T]}$ when $X_t = x \in \mathbb{R}^d$

$$\partial_t u(t,x) + \frac{1}{2}\Delta u(t,x) - \underbrace{\frac{1}{2}|\partial_x u(t,x)|^2}_{\inf_\alpha[\alpha\cdot\partial_x u(t,x) + \frac{1}{2}|\alpha|^2]} + f(x,\mu_t) = 0$$

$$u(T,x) = g(x,\mu_T)$$

- Dynamics of $(\mu_t)_{t\in[0,T]}$
  - Fokker-Planck with optimal feedback is $\alpha^\star(t,x) = -\partial_x u(t,x)$

$$\partial_t \mu_t - \frac{1}{2}\Delta\mu_t - \mathrm{div}(\mu_t \partial_x u(t,x)) = 0 \qquad \begin{cases} t \in [0,T] \\ \mu_0 = \delta_{x_0} \end{cases}$$

  - marginal law of diffusion process

$$dX_t^\star = -\partial_x u(t, X_t^\star)dt + dW_t = -\partial_x \mathcal{U}(t, X_t^\star, \mathcal{L}(X_t^\star))dt + dW_t$$

Connection with the MFG system ($\eta > 0$)

- Regard $\mathcal{U}$ as the generalized value function of the MFG system
  - $\mathcal{U}(t_0, x_0, \mu^0) = u^{\boldsymbol{\mu}:\mu_{t_0}=\mu^0}(t_0, x_0)$
- Optimization in environment $(\mu_t)_{t\in[0,T]} \rightsquigarrow$ HJB equation
  - $u(t,x)$ = minimal cost under $(\mu_t)_{t\in[0,T]}$ when $X_t = x \in \mathbb{R}^d$

$$\partial_t u(t,x) + \frac{1}{2}\Delta u(t,x) - \underbrace{\frac{1}{2}|\partial_x u(t,x)|^2}_{\inf_\alpha[\alpha\cdot\partial_x u(t,x) + \frac{1}{2}|\alpha|^2]} + f(x,\mu_t) = 0$$

$$u(T,x) = g(x,\mu_T)$$

- Dynamics of $(\mu_t)_{t\in[0,T]}$
  - Fokker-Planck with optimal feedback is $\alpha^\star(t,x) = -\partial_x u(t,x)$

$$\partial_t \mu_t - \frac{1}{2}\Delta\mu_t - \mathrm{div}(\mu_t \partial_x u(t,x)) + \sqrt{\eta}\,\mathrm{div}(\mu_t \frac{dB_t}{dt}) = 0$$

  - marginal law of diffusion process

$$dX_t^\star = -\partial_x \mathcal{U}(t, X_t^\star, \mathcal{L}(X_t^\star|B))dt + dW_t + \sqrt{\eta}dB_t$$

Solving the master equation

- Well posedness of $\mathcal{U}$ requires $\exists!$ for MFG system
- Need additional monotonicity condition to prevent shocks
  - Lasry-Lions monotonicity in direction μ (same with g)

$$\int_{\mathbb{R}^d} (f(x,\mu) - f(x,\mu'))d(\mu - \mu')(x) \geq 0$$

  - Example: let L be $\nearrow$ and ρ be even and set

$$h(x,\mu) = \int_{\mathbb{R}^d} L(\rho \star \mu(z))\rho(x-z)dz$$

- Linearization $\leadsto$ differentiability in μ^0 $\leadsto$ use convex perturbation
  - requires smooth coefficients with bounded derivatives

$$\frac{d}{d\varepsilon}_{|\varepsilon=0+} u^{(1-\varepsilon)\mu+\varepsilon\mu'}(t_0,\cdot) = \frac{d}{d\varepsilon}_{|\varepsilon=0+} \mathcal{U}(t_0,\cdot,(1-\varepsilon)\mu+\varepsilon\mu')$$

$$= \int_{\mathbb{R}^d} \mathcal{V}(t_0,\cdot,\mu)(y)d(\mu'-\mu)(y)$$

  - $\partial_y \mathcal{V}(t_0,\cdot,\mu)(y) = \partial_\mu \mathcal{U}(t_0,\cdot,\mu)(y)$

Part III. Convergence

Connection Nash system/master equation

- Now it makes sense to let $u^{N,i}(t,\boldsymbol{x}) = \mathcal{U}(t, x_i, \bar{\mu}^N_{\boldsymbol{x}})$
- Using smoothness of $\mathcal{U}$ at order 2 $\rightsquigarrow$ we show

$$\partial_t u^{N,i}(t,\boldsymbol{x}) + \frac{1}{2}\sum_j \Delta_{x_j} u^{N,i}(t,\boldsymbol{x}) + \frac{\eta}{2}\sum_{j,k} \mathrm{Tr} D^2_{x_j,x_k} u^{N,i}(t,\boldsymbol{x})$$
$$- \sum_{j\neq i} \partial_{x_j} u^{N,i}(t,\boldsymbol{x}) \cdot \partial_{x_j} u^{N,j}(t,\boldsymbol{x})$$
$$- \frac{1}{2}|\partial_{x_i} u^{N,i}(t,\boldsymbol{x})|^2 + \big(f(x_i, \bar{\mu}^N_{\boldsymbol{x}}) + \underbrace{r^{N,i}(t,\boldsymbol{x})}_{|r^{N,i}| \leq C/N}\big) = 0$$

  - with $\bar{x}^N = \frac{1}{N}\sum_{j=1}^N x_j$
- Propagation of reminder $O(1/N)$ among N players?

Comparison of value functions

- Equilibrium trajectories of the N player game

$$dX_t^{N,i} = -\partial_{x_i} v^{N,i}(t, X_t^{N,1}, \cdots, X_t^{N,N})dt + dW_t^i + \sqrt{\eta}dB_t$$

- Value processes

$$Y_t^{N,i} = v^{N,i}(t, X_t^{N,1}, \cdots, X_t^{N,N}), \quad Z_t^{N,i,j} = \partial_{x_j} v^{N,i}(t, X_t^{N,1}, \cdots, X_t^{N,N})$$
$$\hat{Y}_t^{N,i} = u^{N,i}(t, X_t^{N,1}, \cdots, X_t^{N,N}), \quad \hat{Z}_t^{N,i,j} = \partial_{x_j} u^{N,i}(t, X_t^{N,1}, \cdots, X_t^{N,N})$$

- Itô's formula

$$dY_t^{N,i} = -(\frac{1}{2}|Z_t^{N,i,i}|^2 + f(X_t^{N,i}, \bar{\mu}_t^N))dt + \sum_j Z_t^{N,i,j} \cdot (dW_t^j + \sqrt{\eta}dB_t)$$

$$d\hat{Y}_t^{N,i} = -(\frac{1}{2}|\hat{Z}_t^{N,i,i}|^2 + f(X_t^{N,i}, \bar{\mu}_t^N) + r^{N,i}(t, X_t^{N,i}))dt + \sum_j \hat{Z}_t^{N,i,j} \cdot (\hat{Z}_t^{N,j,j} - Z_t^{N,j,j})dt + \sum_j \hat{Z}_t^{N,i,j} \cdot (dW_t^j + \sqrt{\eta}dB_t)$$

with $Y_T^{N,i} = g(X_T^i, \bar{\mu}_T^N)$ and $\hat{Y}_T^{N,i} = g(X_T^i, \bar{\mu}_T^N)$, and $\bar{\mu}_t^N = \frac{1}{N}\sum_j \delta_{X_t^{N,j}}$

Stability argument

- Difference between two dynamics

$$d(\hat{Y}_t^{N,i} - Y_t^{N,i})$$

$$= -\Big[\frac{1}{2}|\hat{Z}_t^{N,i,i}|^2 - \frac{1}{2}|Z_t^{N,i,i}|^2 + \underbrace{r^{N,i}(t, X_t^{N,i})}_{\sim C/N}\Big]dt$$

$$+ \sum_j \underbrace{\hat{Z}_t^{N,i,j}}_{\leq C/N \text{ if } i \neq j} (\hat{Z}_t^{N,j,j} - Z_t^{N,j,j})dt$$

$$+ \sum_j (\hat{Z}_t^{N,i,j} - Z_t^{N,i,j}) \cdot dW_t^j + (\sum_j \hat{Z}_t^{N,i,j} - \sum_j Z_t^{N,i,j}) \cdot \sqrt{\eta} dB_t$$

- Observe that $\hat{Y}_T^{N,i} = Y_T^{N,i}$
  - if no dt terms except $O(1/N)$

$$\hat{Y}_t^{N,i} - Y_t^{N,i}$$

$$+ \int_t^T \sum_j (\hat{Z}_t^{N,i,j} - Z_t^{N,i,j}) \cdot dW_s^j + (\sum_j \hat{Z}_t^{N,i,j} - \sum_j Z_t^{N,i,j}) \cdot \sqrt{\eta} dB_s = O(\frac{1}{N})$$

Stability argument

- Difference between two dynamics

$$
\begin{aligned}
& d(\hat{Y}_t^{N,i} - Y_t^{N,i}) \\
& = -\Big[\frac{1}{2}|\hat{Z}_t^{N,i,i}|^2 - \frac{1}{2}|Z_t^{N,i,i}|^2 + \underbrace{r^{N,i}(t, X_t^{N,i})}_{\sim C/N}\Big]dt \\
& \quad + \sum_j \underbrace{\hat{Z}_t^{N,i,j}}_{\leq C/N \text{ if } i \neq j} (\hat{Z}_t^{N,j,j} - Z_t^{N,j,j})dt \\
& \quad + \sum_j (\hat{Z}_t^{N,i,j} - Z_t^{N,i,j}) \cdot dW_t^j + \Big(\sum_j \hat{Z}_t^{N,i,j} - \sum_j Z_t^{N,i,j}\Big) \cdot \sqrt{\eta} dB_t
\end{aligned}
$$

- Observe that $\hat{Y}_T^{N,i} = Y_T^{N,i}$
  - if no dt terms except $O(1/N)$

$$
\mathbb{E}[|\hat{Y}_t^{N,i} - Y_t^{N,i}|^2]
$$

$$
+ \mathbb{E}\int_t^T \sum_j \Big|\hat{Z}_t^{N,i,j} - Z_t^{N,i,j}\Big|^2 + \eta \mathbb{E}\int_t^T \Big|\sum_j \hat{Z}_t^{N,i,j} - \sum_j Z_t^{N,i,j}\Big|^2 ds = O(\tfrac{1}{N^2})
$$

Stability argument

- Difference between two dynamics

$$
\begin{aligned}
& d(\hat{Y}_t^{N,i} - Y_t^{N,i}) \\
& = -\Big[\frac{1}{2}|\hat{Z}_t^{N,i,i}|^2 - \frac{1}{2}|Z_t^{N,i,i}|^2 + \underbrace{r^{N,i}(t, X_t^{N,i})}_{\sim C/N}\Big]dt \\
& \quad + \sum_j \underbrace{\hat{Z}_t^{N,i,j}}_{\leq C/N \text{ if } i \neq j} (\hat{Z}_t^{N,j,j} - Z_t^{N,j,j})dt \\
& \quad + \sum_j (\hat{Z}_t^{N,i,j} - Z_t^{N,i,j}) \cdot dW_t^j + \Big(\sum_j \hat{Z}_t^{N,i,j} - \sum_j Z_t^{N,i,j}\Big) \cdot \sqrt{\eta} dB_t
\end{aligned}
$$

- Do as if $|\cdot|^2$ is Lipschitz $\rightsquigarrow$ take the square and $\mathbb{E}$

$$
\mathbb{E}\Big[\Big|\hat{Y}_t^{N,i} - Y_t^{N,i}\Big|^2 + \int_t^T \sum_{j=1}^N \Big|\hat{Z}_s^{N,i,j} - Z_s^{N,i,j}\Big|^2 ds\Big]
$$

$$
\leq \frac{C_\epsilon}{N^2} + \epsilon \mathbb{E} \int_t^T \Big|\hat{Z}_s^{N,i,i} - Z_s^{N,i,i}\Big|^2 ds + \frac{\epsilon}{N} \sum_j \mathbb{E} \int_t^T \Big|\hat{Z}_s^{N,j,j} - Z_s^{N,j,j}\Big|^2 ds
$$

Stability argument

- Difference between two dynamics

$$d(\hat{Y}_t^{N,i} - Y_t^{N,i})$$
$$= -\Big[\frac{1}{2}|\hat{Z}_t^{N,i,i}|^2 - \frac{1}{2}|Z_t^{N,i,i}|^2 + \underbrace{r^{N,i}(t, X_t^{N,i})}_{\sim C/N}\Big]dt$$
$$+ \sum_j \underbrace{\hat{Z}_t^{N,i,j}}_{\leq C/N \text{ if } i \neq j} (\hat{Z}_t^{N,j,j} - Z_t^{N,j,j})dt$$
$$+ \sum_j (\hat{Z}_t^{N,i,j} - Z_t^{N,i,j}) \cdot dW_t^j + \Big(\sum_j \hat{Z}_t^{N,i,j} - \sum_j Z_t^{N,i,j}\Big) \cdot \sqrt{\eta} dB_t$$

- To handle the square $\rightsquigarrow$ exponential transform $\Rightarrow$ final result

$$\mathbb{E}\Big[\sup_{0 \leq t \leq T} |\hat{Y}_t^{N,i} - Y_t^{N,i}|^2\Big] + \mathbb{E}\int_0^T |\hat{Z}_t^{N,i,i} - Z_t^{N,i,i}|^2 dt \leq \frac{C}{N^2}$$

- Inserting in the forward equation

$$dX_t^{N,i} = -Z_t^{N,i,i}dt + dW_t^i + \sqrt{\eta} dB_t$$
$$\approx -\hat{Z}_t^{N,i,i}dt + dW_t^i + \sqrt{\eta} dB_t$$

Part IV. Rate of convergence

Fluctuations

- Equilibrium trajectories of the N player game

$$dX_t^{N,i} = -\partial_{x_i} v^{N,i}(t, X_t^{N,1}, \cdots, X_t^{N,N})dt + dW_t^i + \sqrt{\eta} dB_t$$

$$= -\Big[\partial_x \mathcal{U}\Big(t, X_t^{N,i}, \frac{1}{N}\sum_{j=1}^{N} \delta_{X_t^{N,j}}\Big) + O(\tfrac{1}{N})\Big]dt + dW_t^i + \sqrt{\eta} dB_t$$

- Compare with

$$d\hat{X}_t^{N,i} = -\partial_x \mathcal{U}\Big(t, \hat{X}_t^{N,i}, \frac{1}{N}\sum_{j=1}^{N} \delta_{\hat{X}_t^{N,j}}\Big)dt + dW_t^i + \sqrt{\eta} dB_t$$

  - get $\mathbb{E}[\sup_{0\le t\le T} |\hat{X}_t^{N,i} - X_t^{N,i}|^2] \le \dfrac{C}{N^2}$
  - and $\mathbb{E}\Big[\sup_{0\le t\le T} W_2\Big(\frac{1}{N}\sum_{i=1}^{N} \delta_{\hat{X}_t^{N,i}}, \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^{N,i}}\Big)^2\Big] \le \dfrac{C}{N^2}$

- Limit is $\partial_t \mu_t - \frac{1}{2}\Delta \mu_t - \mathrm{div}(\mu_t \partial_x \mathcal{U}(t, \cdot, \mu_t)) + \sqrt{\eta}\,\mathrm{div}(\mu_t \dot{B}_t) = 0$