Stability of mean field model
for opinion dynamics and collective motion

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Opinion dynamics

• Propose an opinion model for an interacting population of agents to study consensus convergence, with simple local rules of interaction.

• Opinion model for \( N \) agents [1]:

\[
dx_i = -\frac{1}{N} \sum_{j=1}^{N} \phi(|x_i - x_j|)(x_i - x_j)dt + \sigma dW^i(t), \quad i = 1, \ldots, N,
\]

where \( x_i(t) \) is the agent \( i \)'s opinion.

• The influence function \( \phi \) is nonnegative, bounded, and compactly supported in \([0, 1]\).

\[ \leftrightarrow \text{The interactions are attractive and the agent } i \text{ is only affected by agents that have similar opinions.} \]

• The initial opinions \( x_i(0) \) may be deterministic or random, for instance \( x_i(0) \) are i.i.d. with distribution with density \( \rho_0 \).

• The independent Brownian motions \( W^i(t), \ i = 1, \ldots, N \) model external noise, and \( \sigma \geq 0 \) is its strength.

Opinion dynamics: Simulations

\begin{align*}
\phi(s) &= 1_{[0,1/\sqrt{2}]}(s) + 0.1 \times 1_{(1/\sqrt{2},1]}(s) \\
\text{Initial uniform distribution over } [0,L], \quad L = 10, \quad N = 500.
\end{align*}

Can we predict the number of clusters?
Opinion dynamics: The mean field limit

• Consider the model in $[0, L]$ with periodic boundary conditions.

• Introduce the empirical probability measure $\rho^N(t, dx)$ of the opinions of the agents:

$$\rho^N(t, dx) = \frac{1}{N} \sum_{j=1}^{N} \delta_{x_j(t)}(dx).$$

$\rho^N(t, dx)$ is a measure-valued stochastic process.

• Assume that as $N \to \infty$, $\rho^N(0, dx)$ converges weakly, in probability, to a deterministic measure with density $\rho_0(x)$.

This happens (with $\rho_0(x) = 1/L$) if the initial opinions are i.i.d. with uniform density over $[0, L]$.

• As $N \to \infty$, $\rho^N(t, dx)$ converges weakly, in probability, to a deterministic probability measure whose density $\rho(t, x)$ satisfies (in a weak sense) the nonlinear Fokker-Planck equation [1]:

$$\frac{\partial \rho}{\partial t}(t, x) = \frac{\partial}{\partial x} \left\{ \left[ \int \rho(t, x-y)y\phi(|y|)dy \right] \rho(t, x) \right\} + \frac{\sigma^2}{2} \frac{\partial^2 \rho}{\partial x^2}(t, x),$$

with initial density $\rho_0(x)$.

Opinion dynamics: The mean field limit

Formally:

\[ dx_i = -\left[ \frac{1}{N} \sum_{j=1}^{N} \phi(|x_i - x_j|)(x_i - x_j) \right] dt + \sigma dW^i(t), \quad i = 1, \ldots, N \]

\[ \Downarrow \]

\[ dx_i = -\left[ \int \phi(|x_i - y|)(x_i - y)\rho^{N}(t, dy) \right] dt + \sigma dW^i(t), \quad i = 1, \ldots, N \]

\[ \Downarrow N \to \infty \]

\[ dX_t = -\left[ \int \phi(|X_t - y|)(X_t - y)\rho(t, y)dy \right] dt + \sigma dW(t), \quad \rho(t, \cdot) = \text{pdf of } X_t \]

\[ \Downarrow \]

\[ \frac{\partial \rho}{\partial t}(t, x) = \frac{\partial}{\partial x} \left\{ \left[ \int \phi(|x - y|)(x - y)\rho(t, y)dy \right] \rho(t, x) \right\} + \frac{\sigma^2}{2} \frac{\partial^2 \rho}{\partial x^2}(t, x) \]

Opinion dynamics for $\sigma = 0$

- Modulational instability (in the mean field limit $N = \infty$).

- Linearize the Fokker-Planck equation by assuming $\rho(t, x) = \rho_0 + \rho_1(t, x)$, $\rho_0 = 1/L$:

  \[ \frac{\partial \rho_1}{\partial t}(t, x) = \rho_0 \int \frac{\partial \rho_1}{\partial x}(t, x - y) y \phi(|y|) dy. \]

- Take the Fourier transform in $x$, $\hat{\rho}_1(t, k) = \int_0^L e^{-ikx} \rho_1(t, x) dx$, with the discrete frequency $k$ in $\mathcal{K} = \{2\pi n/L, n \in \mathbb{N}\}$:

  \[ \frac{\partial \hat{\rho}_1}{\partial t}(t, k) = \left[ i \rho_0 k \int e^{-iky} y \phi(|y|) dy \right] \hat{\rho}_1(t, k). \]

- For each $k$, $|\hat{\rho}_1(t, k)| = |\hat{\rho}_1(0, k)| \exp(\gamma_k t)$, where the growth rate of the $k$-th mode is:

  \[ \gamma_k = \text{Re} \left[ i \rho_0 k \int e^{-iky} y \phi(|y|) dy \right] = \rho_0 k \int \sin(ky) y \phi(|y|) dy. \]

- The growth rate $\gamma_k$ is maximal for $k = \pm k_{\text{max}}$ with

  \[ k_{\text{max}} = \arg \max_{k \in \mathcal{K}} [\psi(k)], \quad \psi(k) = 2k \int_0^1 \sin(k s) \phi(s) s ds \]
Opinion dynamics for $\sigma = 0$

- Fluctuation theory (in the regime $N \gg 1$).
- Denote $\rho_0(dx) = dx/L$. Fix $T$. Consider
  $$\rho_1^N(t, dx) := \sqrt{N} \left( \rho^N(t, dx) - \rho_0(dx) \right), \quad t \in [0, T]$$

- If the initial opinions $(x_j(0))_{j=1}^N$ are i.i.d. with uniform density over $[0, L]$, then, as $N \gg 1$,
  $$\hat{\rho}_1^N(t = 0, k) = \int e^{-ikx} \rho_1^N(t = 0, dx) = \frac{1}{\sqrt{N}} \sum_{j=1}^N e^{-ikx_j(0)}, \quad k \in \{2\pi n/L, n \in \mathbb{N}^*\}$$
  converge to $\hat{\rho}_1(t = 0, k)$ i.i.d. complex circular Gaussian random variables, mean zero, and variance 1:
  $$\mathbb{E}[\hat{\rho}_1(t = 0, k)] = 0, \quad \mathbb{E}\left[\hat{\rho}_1(t = 0, k)\hat{\rho}_1(t = 0, k')\right] = \delta_{kk'}, \quad k, k' \in \{2\pi n/L, n \in \mathbb{N}^*\}.$$  

Note $\hat{\rho}_1^N(t, k = 0) = 0$.

- As $N \gg 1$, $\hat{\rho}_1^N(t, k), k \in K\{0\}$, converge to $\hat{\rho}_1(t, k), k \in K\{0\}$, independent complex circular Gaussian random variables, with mean zero and variance $\exp(2\gamma_k t)$:
  $$\mathbb{E}\left[\hat{\rho}_1(t, k)\hat{\rho}_1(t, k')\right] = \delta_{kk'} \exp(2\gamma_k t), \quad k, k' \in \{2\pi n/L, n \in \mathbb{N}^*\}.$$  

→ The mode with the largest growth rate $\gamma_{\max} = 2\rho_0 \psi(k_{\max})$ quickly dominates.
Opinion dynamics for $\sigma = 0$

- The time up to the onset of clustering is when $\rho_1 \sim \sqrt{N} \rho_0$:

$$t_{clu} \simeq \frac{1}{2\gamma_{\text{max}}} \ln N \simeq \frac{1}{2\rho_0 \psi(k_{\text{max}})} \ln N$$

Clustering happens with a mean distance between clusters equal to $2\pi/k_{\text{max}}$.

- Once clustering has occurred, two types of dynamical evolutions are possible:
  1. If $2\pi/k_{\text{max}} > 1$, then the clusters do not interact with each other because they are beyond the range of the influence function.
     $\leftrightarrow$ The situation is frozen and there is no consensus convergence.
Opinion dynamics for $\sigma = 0$

Here $\phi(s) = 1_{[0,1/\sqrt{2}]}(s) + 0.1 \times 1_{(1/\sqrt{2},1]}(s)$, $L = 10$, $N = 500$.

$k_{\text{max}} = 3.77$. Inter-cluster distance = 1.67.

$\leftrightarrow$ No consensus convergence.
Opinion dynamics for $\sigma = 0$

- The time up to the onset of clustering is when $\rho_1 \sim \sqrt{N}\rho_0$:
  \[
  t_{clu} \simeq \frac{1}{2\gamma_{\text{max}}} \ln N \simeq \frac{1}{2\rho_0\psi(k_{\text{max}})} \ln N
  \]

Clustering happens with a mean distance between clusters equal to $2\pi/k_{\text{max}}$.

- Once clustering has occurred, two types of dynamical evolutions are possible:
  1. If $2\pi/k_{\text{max}} > 1$, then the clusters do not interact with each other because they are beyond the range of the influence function.
     $\leftarrow$ The situation is frozen and there is no consensus convergence.
  2. If $2\pi/k_{\text{max}} < 1$, then the clusters interact with each other.
     $\leftrightarrow$ There may be consensus convergence.
     However, consensus convergence is not guaranteed as clusters may merge by packets, and the centers of the new clusters may be separated by a distance larger than $2\pi/k_{\text{max}}$, and then global consensus convergence does not happen.
     $\leftarrow$ The number of mega-clusters formed by this dynamic is not easy to predict.

Zürich May 2017
Opinion dynamics for $\sigma = 0$

Here $\phi(s) = 0.2 \times 1_{[0, 1/\sqrt{2}]}(s) + 1_{(1/\sqrt{2}, 1]}(s)$, $L = 10$, $N = 500$.

$k_{\text{max}} = 9.42$. Inter-cluster distance $= 0.67$.

$\twoheadrightarrow$ There may be global consensus convergence (very sensitive!).

Zürich May 2017
Opinion dynamics for $\sigma > 0$

- Modulational instability (in the mean field limit $N = \infty$).

- Linearize the Fokker-Planck equation by assuming $\rho(t, x) = \rho_0 + \rho_1(t, x)$, $\rho_0 = 1/L$:
  \[
  \frac{\partial \rho_1}{\partial t}(t, x) = \rho_0 \int \frac{\partial \rho_1}{\partial x}(t, x - y)y\phi(|y|)dy + \frac{\sigma^2}{2} \frac{\partial^2 \rho_1}{\partial x^2}(t, x).
  \]

- In the Fourier domain:
  \[
  \frac{\partial \hat{\rho}_1}{\partial t}(t, k) = \left[ i\rho_0 k \int e^{-iky}y\phi(|y|)dy - \frac{\sigma^2 k^2}{2} \right] \hat{\rho}_1(t, k).
  \]

- Growth rates of the modes:
  \[
  \gamma_{\sigma, k} = \text{Re} \left[ i\rho_0 k \int e^{-iky}y\phi(|y|)dy - \frac{\sigma^2 k^2}{2} \right],
  \]
  or $\gamma_{\sigma, k} = \rho_0 \psi_{\sigma}(k)$, where
  \[
  \psi_{\sigma}(k) = 2k \int_0^1 \sin(ky)y\phi(|y|)dy - \frac{\sigma^2 k^2}{2\rho_0}.
  \]

We look for the most unstable mode whose frequency $k_{\text{max}} \in \{2\pi n/L, n \in \mathbb{N}\}$ maximizes $\psi_{\sigma}(k)$.
Opinion dynamics for $\sigma > 0$

Let

$$\sigma_c^2 = \max_{k \in K} \left[ \frac{4 \rho_0}{k} \int_0^1 \phi(s) s \sin(k s) ds \right] \simeq 4 \rho_0 \int_0^1 s^2 \phi(s) ds.$$ 

1. If $\sigma < \sigma_c$, then $\max_{k>0} \psi_{\sigma}(k) > 0$, so $k_{\text{max}} = \argmax_{k>0} \psi_{\sigma}(k) > 0$ exists and $\hat{\rho}_1(t, k_{\text{max}})$ has positive growth rate $\gamma_{\text{max}} = \rho_0 \psi_{\sigma}(k_{\text{max}})$.  

$\rightarrow$ The system is linearly unstable (qualitatively analogous to the deterministic case, although $k_{\text{max}}$ is reduced).

2. If $\sigma > \sigma_c$, then $\max_{k>0} \psi_{\sigma}(k) < 0$, so all of $\hat{\rho}_1(t, k)$ have negative growth rates. 

$\leftarrow$ The system with uniform density is stable.
Opinion dynamics for $\sigma > 0$

• Fluctuation theory (in the regime $N \gg 1$).

• The measure-valued process

$$\rho_1^N(t, dx) := \sqrt{N} \left( \rho^N(t, dx) - \rho_0(dx) \right)$$

converges in distribution as $N \to \infty$ to a measure-valued process $\rho_1(t, dx)$ whose density $\rho_1(t, x)$ satisfies a stochastic PDE:

$$d\rho_1(t, x) = \left[ \rho_0 \int \frac{\partial \rho_1}{\partial x}(t, x - y)y\phi(|y|)dy + \frac{\sigma^2}{2} \frac{\partial^2 \rho_1}{\partial x^2}(t, x) \right] dt + \sigma dW(t, x)$$

where $W(t, x)$ is a space-time Gaussian random noise with mean zero and covariance

$$\text{Cov} \left( \int_0^L W(s, x)f_1(x)dx, \int_0^L W(t, x)f_2(x)dx \right) = \frac{\min\{s, t\}}{L} \int_0^L f'_1(x)f'_2(x)dx$$

for any test functions $f_1(x)$ and $f_2(x)$, which is independent of the Gaussian (white noise) initial condition.

• $\hat{\rho}_1(t, k)$ is a Gaussian process with mean zero and covariance

$$\mathbb{E} \left[ \hat{\rho}_1(t, k)\hat{\rho}_1(t, k') \right] = \delta_{kk'} \left\{ \exp(2\gamma_{\sigma, k}t) \left[ 1 + \frac{\sigma^2k^2}{2\gamma_{\sigma, k}} \right] - \frac{\sigma^2k^2}{2\gamma_{\sigma, k}} \right\}, \quad k, k' \in \{2\pi n/L, n \in \mathbb{N}\}.$$ 

→ The mode with the largest growth rate $\gamma_{\sigma, k_{\max}} = \rho_0\psi_\sigma(k_{\max})$ quickly dominates.
Opinion dynamics for $\sigma > 0$

$\phi(s) = 1_{[0,1]}(s), \sigma = 0.1$

$N = 500$

Inter-Cluster Distance = 2.5$

Opinions, $x_i(t)$

Simulation
Opinion dynamics for $\sigma > 0$

\[ \phi(s) = 1_{[0,1]}(s), \sigma = 0.2 \]

\[ N = 500 \]

Inter-Cluster Distance = 3.33

\[ \text{Simulation} \]
**Opinion dynamics for** \( \sigma > 0 \)

\[
\phi(s) = 1_{[0,1]}(s), \sigma = 0.365
\]

\[
\psi_\sigma(k)
\]

Simulation

\[
\phi(s) = 1_{[0,1]}(s), \sigma_c = 0.36515, \sigma = 0.36515
\]

\[
N = 500
\]

Opinions, \( x_i(t) \)
Opinion dynamics for $\sigma > 0$

$\psi_\sigma(k)$

$\phi(s) = 1_{[0,1]}(s), \sigma = 0.5$

$N = 500$

Simulation
Opinion dynamics for $\sigma > 0$

After clustering:

- Markovian dynamics of clusters (move like Brownian motions and merge when two of them come close to each other).

- Global consensus convergence (in 1D, Brownian motions always collide).

- Megacluster moves like a Brownian motion.
Collective motion
Collective motion: Czirók model

• $N$ agents move along the torus $[0, L]$.

• For $i = 1, \ldots, N$, the position $x_i$ and velocity $u_i$ of particle $i$ satisfy:

$$dx_i = u_i dt,$$
$$du_i = \left[G\left(\langle u \rangle_i\right) - u_i\right] dt + \sigma dW_i(t).$$

• $\{W_i(t)\}_{i=1}^N$ are independent Brownian motions.

• $\langle u \rangle_i$ is a weighted average of the velocities $\{u_j\}_{j=1}^N$:

$$\langle u \rangle_i = \frac{1}{N} \sum_{j=1}^N u_j \phi(|x_j - x_i|),$$

with the weights depending on the distance (on the torus) between the position $x_i$ and the positions of the other agents.

$\phi(x)$ is a nonnegative influence function normalized so that $\frac{1}{L} \int_0^L \phi(|x|)dx = 1$.

• $G(u)$ is an odd and smooth function.

If $G(u) = u$: Cucker-Smale model.

If $G(u)$ derives from a double well potential: Czirók model.
Collective motion bistability: Experiments

Locusts in a torus  Alignment as function of time

Collective motion: Simulations

\[ \phi(s) = 5 \times 1_{[0,1]}(s). \]

\[ G(u) = 2u - 0.072u^3. \]

\[ \sigma = 5. \]

Initial distribution: \((x_i(0), u_i(0))_{i=1}^N\) i.i.d., \((x_i(0))_{i=1}^N\) uniform over \([0, L]\), \((u_i(0))_{i=1}^N\) Gaussian distributed with mean zero and variance \(\sigma^2/2\), \(L = 10\), \(N = 100\).

Can we explain the bistability, the transition rate?
Collective motion: The mean field limit

• Introduce the empirical probability measure:

\[ \rho_N(t, dx, du) = \frac{1}{N} \sum_{i=1}^{N} \delta(x_i(t), u_i(t)) \, (dx, du). \]

• If \( \rho_N(0, dx, du) \) converges to a deterministic measure \( \bar{\rho}(x, u) \, dx \, du \) as \( N \to \infty \), then as \( N \to \infty \), \( \rho_N(t, dx, du) \) converges to the deterministic measure \( \rho(t, x, u) \, dx \, du \) whose density is the solution of the nonlinear Fokker-Planck equation

\[
\frac{\partial \rho}{\partial t} = -u \frac{\partial \rho}{\partial x} - \frac{\partial}{\partial u} \left\{ G \left( \int \int u' \phi(|x'|) \rho(t, x - x', u') \, du' \, dx' \right) - u \right\} \rho + \frac{1}{2} \sigma^2 \frac{\partial^2 \rho}{\partial u^2},
\]

starting from \( \rho(t = 0, x, u) = \bar{\rho}(x, u) \).
Collective motion: The stationary states in the mean field limit

- The stationary solutions $\rho(x, u)$ have the following form:

  $$\rho_\xi(x, u) = \frac{1}{L} F_\xi(u), \quad F_\xi(u) = \frac{1}{\sqrt{\pi \sigma^2}} \exp \left( - \frac{(u - \xi)^2}{\sigma^2} \right).$$

- They are uniform in space, Gaussian in velocity, and their mean velocity $\xi$ satisfies the compatibility condition:

  $$\xi = G(\xi).$$

- There are, therefore, as many stationary equilibria as there are solutions to the compatibility equation.

- When $G$ is such that $u - G(u)$ derives from a double-well potential, such as $G(u) = 2 \tanh(u)$ or $G(u) = 2u - u^3$, there are three $\xi$ satisfying the compatibility condition: 0 and $\pm \xi_e$, with $\xi_e > 0$. 
Collective motion: Linear stability analysis for the stationary states

- Let $\xi$ be such that $G(\xi) = \xi$ and consider
  \[ \rho(t, x, u) = \rho_{\xi}(x, u) + \rho^{(1)}(t, x, u) = \frac{1}{L} F_{\xi}(u) + \rho^{(1)}(t, x, u), \]
  for small perturbation $\rho^{(1)}$. By linearizing the nonlinear Fokker-Planck equation:
  \[
  \frac{\partial \rho^{(1)}}{\partial t} = -u \frac{\partial \rho^{(1)}}{\partial x} - \frac{\partial}{\partial u} \left[ (\xi - u)\rho^{(1)} \right] \\
  - \frac{1}{L} G'(\xi) \left[ \int \int u' \phi(|x'|) \rho^{(1)}(t, x - x', u') du' dx' \right] F'_{\xi}(u) + \frac{1}{2} \sigma^2 \frac{\partial^2 \rho^{(1)}}{\partial u^2}.
  \]

- The mode $\rho^{(1)}_k(t, u) = \frac{1}{L} \int_0^L \rho^{(1)}(t, x, u)e^{i2\pi k x/L} dx$ satisfies
  \[
  \frac{\partial \rho^{(1)}_k}{\partial t} = \frac{i2\pi k}{L} u \rho^{(1)}_k - \frac{\partial}{\partial u} \left[ (\xi - u)\rho^{(1)}_k \right] \\
  - G'(\xi) \phi_k \left[ \int u' \rho^{(1)}_k(t, u') du' \right] F'_{\xi}(u) + \frac{1}{2} \sigma^2 \frac{\partial^2 \rho^{(1)}_k}{\partial u^2},
  \]
  with $\phi_k = \frac{1}{L} \int_0^L \phi(|x|)e^{i2\pi k x/L} dx$.
- The equations are uncoupled in $k$ and the system is linearly stable if all modes are stable.
Collective motion: Linear stability analysis for the stationary states

- The 0-th order mode is stable if and only if $G'(\xi) < 1$.
- The $k$-th order modes are stable if $\sigma$ is large enough (critical $\sigma$ depends on $\phi$, $G$).

When $G$ is such that $u - G(u)$ derives from a double-well potential:

1. the order states

   $$\rho_{\pm\xi_e}(x, u) = \frac{1}{L} \frac{1}{\sqrt{\pi\sigma^2}} \exp \left( -\frac{(u \mp \xi_e)^2}{\sigma^2} \right), \quad \xi_e = G(\xi_e) > 0,$$

   are stable equilibria,

2. the disorder state

   $$\rho_0(x, u) = \frac{1}{L} \frac{1}{\sqrt{\pi\sigma^2}} \exp \left( -\frac{u^2}{\sigma^2} \right)$$

   is an unstable equilibrium.

$\leftrightarrow$ The noise strength $\sigma$ improves the stability of the order states $\rho_{\pm\xi_e}(x, u)$.
Collective motion: Fluctuation analysis

• Fluctuation analysis \((N \gg 1)\).

• Let \(\rho_N(t, dx, du) = \frac{1}{N} \sum_{i=1}^{N} \delta(x_i(t), u_i(t))(dx, du)\) and \(\xi\) be a solution to \(\xi = G(\xi)\).

• If as \(N \to \infty\), \(\rho_N(0, dx, du)\) converges to the stationary state \(\rho_\xi(x, u)dxdu\), then as \(N \to \infty\), \(\rho_N^{(1)}(t, dx, du) = \sqrt{N}[\rho_N(t, dx, du) - \rho_\xi(x, u)dxdu]\) converges to the measure-valued process \(\rho^{(1)}(t, dx, du)\) satisfying

\[
\frac{d\rho^{(1)}}{dt} = -u \frac{\partial \rho^{(1)}}{\partial x} dt - \frac{\partial}{\partial u} [(G(\xi) - u)\rho^{(1)}] dt
\]

\[
- G'(\xi) \frac{\partial \rho_\xi}{\partial u} \left[ \int_0^L \int_{-\infty}^\infty u' \phi(|x' - x|) \rho^{(1)}(t, dx', du') \right] dt + \frac{1}{2} \sigma^2 \frac{\partial^2 \rho^{(1)}}{\partial u^2} dt + \sigma dW_\xi,
\]

where \(W_\xi(t, x, u)\) is a space-time Gaussian random noise with covariance

\[
\text{Cov} \left( \int_0^L \int_{-\infty}^\infty W_\xi(t, x, u) f_1(x, u) dx du, \int_0^L \int_{-\infty}^\infty W_\xi(t', x, u) f_2(x, u) dx du \right)
\]

\[
= \min(t, t') \int_0^L \int_{-\infty}^\infty \frac{\partial f_1}{\partial u}(x, u) \frac{\partial f_2}{\partial u}(x, u) \rho_\xi(x, u) dx du,
\]

for any test functions \(f_1\) and \(f_2\).

• Stability is ensured iff \(\rho_\xi\) is stable for the linearized Fokker-Planck equation.
Collective motion: Simulations

$G(u) = \frac{h+1}{5}u - \frac{h}{125}u^3$, $h = 6$, $\sigma = 0.5$, $N = 1000$.

- Here $G'(\xi) < 1$, so that the mode 0 is stable $\rightarrow$ the average velocity is stable.
- The first mode $k_{\text{max}} = 1$, with growth rate $\gamma(k_{\text{max}}) = \gamma_r(k_{\text{max}}) + i\gamma_i(k_{\text{max}}) \in \mathbb{C}$, is the most unstable.
  $\rightarrow$ a spatial modulation is growing, which gives one cluster.
- The imaginary part of the growth rate gives the velocity of the cluster $v = \frac{\gamma_i(k_{\text{max}})L}{2\pi k_{\text{max}}}$:

$$\exp[-i2\pi k_{\text{max}}x/L] \exp[(\gamma_r(k_{\text{max}}) + i\gamma_i(k_{\text{max}}))t] = \exp[-i(2\pi k_{\text{max}}/L)(x-vt)] \exp[\gamma_r(k_{\text{max}})t]$$
Collective motion: Simulations

\[ G(u) = \frac{h+1}{5} u - \frac{h}{125} u^3, \ h = 6, \ \sigma = 0.5, \ N = 1000. \]

- Here \( G'(\xi) < 1 \), so that the mode 0 is stable \( \rightarrow \) the average velocity is stable.
- The first mode \( k_{\text{max}} = 1 \), with growth rate \( \gamma(k_{\text{max}}) = \gamma_r(k_{\text{max}}) + i\gamma_i(k_{\text{max}}) \in \mathbb{C} \), is the most unstable.
  \( \rightarrow \) a spatial modulation is growing, which gives one cluster.
- The imaginary part of the growth rate gives the velocity of the cluster \( v = \frac{\gamma_i(k_{\text{max}}) L}{2\pi k_{\text{max}}} \):
\[
\exp[-i2\pi k_{\text{max}} x/L] \exp[(\gamma_r(k_{\text{max}}) + i\gamma_i(k_{\text{max}}))t] = \exp[-i(2\pi k_{\text{max}}/L)(x-vt)] \exp[\gamma_r(k_{\text{max}})t]
\]
- The velocity of the cluster is larger than the average velocity of the particles.
  The slow particles of the cluster are left behind!
Collective motion: Large deviations analysis

- Assume that two order states $\rho_{\pm \xi_e}$ exist and are stable.
- Assume initial conditions are such that $\rho(0, x, u) \simeq \rho_{+\xi_e}(x, u)$.
- Consider the rare event:
  \[ A = \{ \rho : \| \rho(T, dx, dy) - \rho_{-\xi_e}(x, u)dxdu \| \leq \delta \} . \]
- We have
  \[ P(\rho_N \in A) \approx \exp \left( -N \inf_{\rho \in A} I(\rho) \right) . \]

The rate function:

\[ I(\rho) = \frac{1}{2\sigma^2} \int_0^T \sup_{f(x,u) : \langle \rho(t,\cdot,\cdot), (\frac{\partial f}{\partial u})^2 \rangle \neq 0} \frac{\langle \frac{\partial \rho}{\partial t} (t,\cdot,\cdot) - \mathcal{L}_\rho^* \rho(t,\cdot,\cdot), f \rangle^2}{\langle \rho(t,\cdot,\cdot), (\frac{\partial f}{\partial u})^2 \rangle} dt , \]

where $\mathcal{L}_\nu^*$ is the differential operator associated to the Fokker-Planck equation:

\[ \mathcal{L}_\nu^* \rho = -u \frac{\partial \rho}{\partial x} - \frac{\partial}{\partial u} \left[ \left( G \left( \int \int u' \phi(|x'|) \nu(x - x', u') du' dx' \right) - u \right) \rho \right] + \frac{1}{2} \sigma^2 \frac{\partial^2 \rho}{\partial u^2} . \]
Empirical average velocity $\bar{u}^n$. Here $h = 6$ and $\sigma = 5$, with $G(u) = \frac{h+1}{5}u - \frac{h}{125}u^3$.

Stability increases with $N$. 
Empirical average velocity \( \bar{u}_n \). Here \( N = 100 \) and \( h = 6 \), with \( G(u) = \frac{h+1}{5}u - \frac{h}{125}u^3 \).

Stability decreases with \( \sigma \) (but provided \( \sigma \) is large enough!).
Empirical average velocity $\bar{u}_n$. Here $N = 100$ and $\sigma = 5$, with $G(u) = \frac{h+1}{5}u - \frac{h}{125}u^3$. Stability increases with $h$. 

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Conclusions

- Linear stability analysis of the nonlinear Fokker-Planck equation of the mean field model gives a lot of insight into the dynamics of the interacting system.
- Noise globally increases the stability (for opinion dynamics and collective motion).
- Simple systems can give complex behaviors.