Stationary solutions to the compressible Navier–Stokes system driven by stochastic forces

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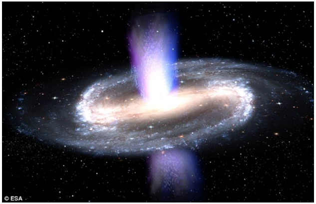
based on joint works with D. Breit, E. Feireisl and B. Maslowski

Maryland, April 2017

Compressible fluids

- fluids having significant changes in fluid density - gas dynamics

Supersonic aircraft breaking the sound barrier



Space wind around a supermassive black hole

Tropical cyclone - Hurricane Fran, 1996

Micro-climate effects of wind turbines

Stochastic NSE for compressible fluids

- time evolution of density ϱ and velocity $\mathbf{u}$ given by

$$\mathrm{d}\varrho + \operatorname{div}(\varrho\mathbf{u})\mathrm{d}t = 0$$
$$\mathrm{d}(\varrho\mathbf{u}) + \big[\operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla\varrho^\gamma\big]\mathrm{d}t = \operatorname{div}\mathbb{S}(\nabla\mathbf{u})\mathrm{d}t + \mathbb{G}(\varrho,\varrho\mathbf{u})\,\mathrm{d}W$$

- with the standard Newtonian viscous stress tensor

$$\mathbb{S}(\nabla\mathbf{u}) = \mu\Big(\nabla\mathbf{u} + \nabla^t\mathbf{u} - \frac{2}{3}\operatorname{div}\mathbf{u}\,\mathbb{I}\Big) + \lambda\operatorname{div}\mathbf{u}\,\mathbb{I}$$

- adiabatic constant $\gamma > \frac{3}{2}$, viscosities $\mu > 0$, $\lambda \geq 0$
- ϱ^γ pressure; $\varrho\mathbf{u}$ momentum

Stochastic perturbation

- W is a cylindrical Wiener process in $\mathfrak{U}$:

$$W(t) = \sum_{k\geq 1} W_k(t) e_k$$

- $\mathbb{G}(\varrho, \varrho\mathbf{u}) = \{\mathbf{G}_k(x, \varrho, \varrho\mathbf{u})\}_{k\geq 1}$ takes values in $\ell^2(L^1(\mathbb{T}^3))$

$$\mathbb{G}(\varrho, \varrho\mathbf{u})\,\mathrm{d}W = \sum_{k\geq 1} \mathbf{G}_k(x, \varrho, \varrho\mathbf{u})\,\mathrm{d}W_k = \sum_{k\geq 1} \varrho\,\mathbf{F}_k(x, \varrho, \varrho\mathbf{u})\,\mathrm{d}W_k$$

  with suitable assumptions on $\mathbf{F}_k$

$\Rightarrow$ $\mathbb{G}(\varrho, \varrho\mathbf{u})$ takes values in $L_2(\mathfrak{U}; W^{-b,2}(\mathbb{T}^3))$, $b > \frac{3}{2}$

Known results

- Tornatore '00, Feireisl, Maslowski, Novotný '13 - weak solutions for $\mathbb{G}(\varrho, \varrho\mathbf{u}) = \varrho\,\mathbb{G}(x)$ via a semi-deterministic approach
- Breit, H. '14 - weak solutions for a general $\mathbb{G}$
- Wang, Wang '15, Smith '15 - Dirichlet boundary conditions
- Breit, Feireisl, H. '15 - incompressible limit
- Breit, Feireisl, H. '15 - relative energy inequality (inviscid–incompressible limit, weak–strong uniqueness)
- Breit, Feireisl, H. '16 - local strong solutions
- Breit, Feireisl, H., Maslowski '17 - stationary solutions

The solution concept – dissipative martingale solutions

- weak solutions in PDE & probabilistic sense
- energy inequality

Dissipative martingale solution

- Λ a probability measure on $L^\gamma(\mathbb{T}^3) \times L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3)$

Then $\big((\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P}), \varrho, \mathbf{u}, W\big)$ is a *dissipative martingale solution* with the initial law Λ

provided

- $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$ is a stochastic basis with (UC)
- W is an $(\mathscr{F}_t)$-cylindrical Wiener process
- $\Lambda = \mathbb{P} \circ \big(\varrho(0), \varrho\mathbf{u}(0)\big)^{-1}$

Dissipative weak martingale solution

- density $\varrho \geq 0$, $\varrho \in C_w([0,T]; L^\gamma(\mathbb{T}^3))$ a.s. and

$$\mathbb{E} \sup_{t\in[0,T]} \|\varrho(t)\|_{L^\gamma}^{\gamma p} < \infty \quad \text{for some} \quad p \in (1,\infty)$$

- velocity field $\mathbf{u} \in L^2(\Omega; L^2(0,T; W^{1,2}(\mathbb{T}^3)))$ satisfies

$$\mathbb{E}\bigg(\int_0^T \|\mathbf{u}\|_{W^{1,2}}^2 \, \mathrm{d}t \bigg)^p \quad \text{for some} \quad p \in (1,\infty)$$

- momentum $\varrho\mathbf{u} \in C_w([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3))$ a.s. and

$$\mathbb{E} \sup_{t\in[0,T]} \|\varrho\mathbf{u}(t)\|_{L^{\frac{2\gamma}{\gamma+1}}}^{\frac{2\gamma}{\gamma+1}p} < \infty \quad \text{for some} \quad p \in (1,\infty)$$

- the continuity eq satisfied in weak and renormalized sense
- the momentum eq satisfied in the weak sense

Energy inequality

- for all $\phi \in C_c^\infty([0,T))$, $\phi \geq 0$, the following energy inequality holds true $\mathbb{P}$-a.s.

$$
\begin{aligned}
&-\int_0^T \partial_t \phi \int_{\mathbb{T}^3} \Big[\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{\varrho}{\gamma-1}\Big]\mathrm{d}x\,\mathrm{d}t + \int_0^T \phi \int_{\mathbb{T}^3} \mathbb{S}(\nabla\mathbf{u}) : \nabla\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leq \phi(0)\int_{\mathbb{T}^3} \Big[\frac{1}{2}\frac{|(\varrho\mathbf{u})(0)|^2}{\varrho(0)} + \frac{\varrho(0)}{\gamma-1}\Big]\mathrm{d}x \\
&\qquad + \sum_{k=1}^\infty \int_0^T \phi \int_{\mathbb{T}^3} \mathbf{G}_k(\varrho,\varrho\mathbf{u})\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}W_k \\
&\qquad + \frac{1}{2}\int_0^T \phi \int_{\mathbb{T}^3} \sum_{k=1}^\infty \varrho^{-1}|\mathbf{G}_k(\varrho,\varrho\mathbf{u})|^2\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

What is the right notion of stationarity?

The notion of stationarity

- no uniqueness – the concept of invariant measures ambiguous

- density ϱ and momentum $\varrho\mathbf{u}$ are stochastic processes

- velocity $\mathbf{u} \in L^2(\Omega; L^2(0,T; W^{1,2}(\mathbb{T}^3)))$

$\Rightarrow$ not a stochastic process in the classical sense

Stationarity vs. weak stationarity

Definition (Stationary stochastic process)

Let $\mathbf{U} = \{\mathbf{U}(t); t \in [0,\infty)\}$ be an X-valued stochastic process. We say that $\mathbf{U}$ is *stationary* provided the joint laws

$$\mathcal{L}(\mathbf{U}(t_1+\tau),\dots,\mathbf{U}(t_n+\tau)), \quad \mathcal{L}(\mathbf{U}(t_1),\dots,\mathbf{U}(t_n))$$

on X^n coincide for all $\tau \geq 0$, for all $t_1,\dots,t_n \in [0,\infty)$.

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Definition (Weakly stationary random variable)

Let $\mathbf{U} : \Omega \to \mathcal{D}'((0,\infty)\times\mathbb{T}^3)$ be weakly measurable. Let $\mathcal{S}_\tau$ be the time shift on the space of trajectories given by $\mathcal{S}_\tau\varphi(t) = \varphi(t+\tau)$. We say that $\mathbf{U}$ is *weakly stationary* provided the laws

$$\mathcal{L}\left(\langle \mathbf{U}, \mathcal{S}_{-\tau}\varphi_1\rangle,\dots,\langle \mathbf{U}, \mathcal{S}_{-\tau}\varphi_n\rangle\right),\quad \mathcal{L}\left(\langle \mathbf{U}, \varphi_1\rangle,\dots,\langle \mathbf{U}, \varphi_n\rangle\right)$$

on $\mathbb{R}^n$ coincide for all $\tau \geq 0$ and all $\varphi_1,\dots,\varphi_n \in C_c^\infty((0,\infty)\times\mathbb{T}^3)$.

Properties

- weak stationarity stable under weak convergence

- weak stationarity of $\mathbf{u} \in L^2_{\mathrm{loc}}(0,\infty; W^{1,2}(\mathbb{T}^3))$ a.s.

$\Rightarrow$ $\mathcal{L}(\mathbf{u}) = \mathcal{L}(\mathcal{S}_\tau \mathbf{u})$ on $L^2_{\mathrm{loc}}(0,\infty; W^{1,2}(\mathbb{T}^3))$

$\Rightarrow$ $\mathcal{L}(\mathbf{u}(s)) = \mathcal{L}(\mathbf{u}(t))$ on $W^{1,2}(\mathbb{T}^3)$ for a.e. $s, t \in [0,\infty)$

- weak stationarity of $\varrho \in C_{\mathrm{loc}}\big([0,\infty); (L^\gamma(\mathbb{T}^3), w)\big)$ a.s.

$\Rightarrow$ ϱ is a stationary $L^\gamma(\mathbb{T}^3)$-valued stochastic process

- weak stationarity of $\varrho\mathbf{u} \in C_{\mathrm{loc}}\big([0,\infty); (L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3), w)\big)$ a.s.

$\Rightarrow$ $\varrho\mathbf{u}$ is a stationary $L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3)$-valued stochastic process

Stationary solutions

Definition

A dissipative martingale solution $[\varrho, \mathbf{u}, W]$ is called *stationary* provided the joint law of the time shift $[\mathcal{S}_\tau \varrho, \mathcal{S}_\tau \mathbf{u}, \mathcal{S}_\tau W - W(\tau)]$ on

$$L^p_{\mathrm{loc}}(0,\infty; L^\gamma(\mathbb{T}^3)) \times L^2_{\mathrm{loc}}(0,\infty; W^{1,2}(\mathbb{T}^3)) \times C_{\mathrm{loc}}([0,\infty); \mathfrak{U}_0)$$

is independent of $\tau \geq 0$, for all $p \in [1,\infty)$.

Theorem (Breit, Feireisl, H., Maslowski '17)

Let the total mass be given by $M_0 \in (0, \infty)$, *that is,*

$$M_0 = \int_{\mathbb{T}^3} \varrho(t, x) \, \mathrm{d}x \quad \textit{for all} \quad t \in (0, \infty).$$

Then there exists a stationary dissipative martingale solution $[\varrho, \mathbf{u}, W]$ *satisfying complete slip boundary conditions.*

A few words about the proof

Four layer approximation scheme

- χ smooth, nonincreasing, $\chi \equiv 1$ on $(-\infty, 0]$, $\chi \equiv 0$ on $[1, \infty)$
- artificial viscosity - ε
- artificial pressure in the momentum equation - δ

$$\mathrm{d}\varrho + \operatorname{div}(\varrho \mathbf{u})\mathrm{d}t = \varepsilon \Delta \varrho \,\mathrm{d}t - 2\varepsilon \varrho \,\mathrm{d}t + \chi\left(\frac{1}{M_0}\int_{\mathbb{T}^3} \varrho \,\mathrm{d}x\right)\mathrm{d}t$$

$$\mathrm{d}(\varrho \mathbf{u}) + \big[\operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla \varrho^\gamma + \delta \nabla \varrho^\beta - \varepsilon \Delta(\varrho \mathbf{u})\big]\mathrm{d}t = \operatorname{div} \mathbb{S}(\nabla \mathbf{u})\mathrm{d}t + \mathbb{G}(\varrho, \varrho \mathbf{u})\,\mathrm{d}W$$

- Faedo-Galerkin finite-dimensional approximation - N
- stopping time argument - R

Aim: $R \to \infty$, $N \to \infty$, $\varepsilon \to 0$, $\delta \to 0$

Four layer approximation scheme

$$\mathrm{d}\varrho + \mathrm{div}(\varrho\mathbf{u})\mathrm{d}t = \varepsilon\Delta\varrho\,\mathrm{d}t - 2\varepsilon\varrho\,\mathrm{d}t + \chi\left(\frac{1}{M_0}\int_{\mathbb{T}^3}\varrho\,\mathrm{d}x\right)\mathrm{d}t$$

$$\mathrm{d}(\varrho\mathbf{u}) + \big[\mathrm{div}(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla\varrho^\gamma + \delta\nabla\varrho^\beta - \varepsilon\Delta(\varrho\mathbf{u})\big]\mathrm{d}t = \mathrm{div}\,\mathbb{S}(\nabla\mathbf{u})\mathrm{d}t + \mathbb{G}(\varrho,\varrho\mathbf{u})\,\mathrm{d}W$$

$+$ Faedo-Galerkin (N) and stopping times (R)

- existence of an invariant measure on the basic level
  – Krylov-Bogoliubov method
- new global-in-time estimates needed
- stationarity preserved under limit procedures

Additional difficulties in comparison to existence

- global-in-time estimates not controlled by the initial data
- new estimates established at every approximation step
- generalized energy inequality needed
- modified method of effective viscous flux
- if $\mathbb{G}(\varrho, \varrho\mathbf{u})\mathrm{d}W \rightsquigarrow \varrho\mathbf{f}(x)\,\mathrm{d}t$, global bounds only for $\gamma > \frac{5}{3}$

Thank you for your attention!