Hydrodynamic flocking model with external potential forcing

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Joint work with Eitan Tadmor

October 25, 2018
The model

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) &= 0 \\
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} &= \int \phi(||\mathbf{x} - \mathbf{y}||)(\mathbf{u}(\mathbf{y}, t) - \mathbf{u}(\mathbf{x}, t))\rho(\mathbf{y}) \, d\mathbf{y} - \nabla \Psi(\mathbf{x})
\end{align*}
\]

- \( \rho, \mathbf{u} \): density and velocity field of a continuum of agents
- \( \phi \): Cucker-Smale interaction potential
- \( \Psi \): external potential

Hydrodynamic Cucker-Smale type flocking model with external potential forcing

September 27, 2018

Abstract
to be written

I used \( C \) kitty, \( C \) meow, \( C \) mi just to distinguish the important constants (if I use \( C_1, C_2, C_3, \) it will be easier to mess up...) We can change them to any reasonable names later.
Motivation

- Particle Cucker-Smale model
  \[
  \begin{aligned}
  \dot{x}_i &= v_i \\
  \dot{v}_i &= \frac{1}{N} \sum_{j \neq i} \phi(\|x_i - x_j\|)(v_j - v_i) \\
  \end{aligned}
  \]
  \(i = 1, \ldots, N\)

- **Flocking**: under suitable assumptions on \(\phi\), there holds
  \[
  \|v_i(t) - v_j(t)\| \to 0, \quad \text{as } t \to \infty
  \]

- Pairwise interaction \(\to\) global velocity alignment

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Cucker-Smale (2007), Ha-Tadmor (2008), Ha-Liu (2009)
Motivation

particle model

\[
\begin{align*}
\dot{x}_i &= v_i \\
\dot{v}_i &= \frac{1}{N} \sum_{j \neq i} \phi(\|x_i - x_j\|)(v_j - v_i) 
\end{align*}
\]

mean field limit

kinetic model

\[
\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot \left( f \int \phi(\|x - y\|)(w - v) f(y, w, t) \, dw \, dy \right) = 0
\]

mono-kinetic ansatz

\[
f(x, v, t) = \rho(x, t) \delta(v - u(x, t))
\]

hydrodynamic model

\[
\begin{align*}
\partial_t \rho + \nabla \cdot (\rho u) &= 0 \\
\partial_t u + u \cdot \nabla u &= \int \phi(\|x - y\|)(u(y, t) - u(x, t)) \rho(y) \, dy
\end{align*}
\]
Motivation

• Another aspect for the hydrodynamic model: existence of global smooth solutions — critical thresholds.

• Cucker-Smale interaction tends to suppress the finite time blow-up of the pressureless Euler equations.


• 2d: Tadmor-Tan (2014), He-Tadmor (2017)
Motivation

- In reality, moving agents are subject to pairwise interaction forces as well as external forces.

- These forces may compete with the alignment forces, make it harder to achieve the flocking state.

- In this talk we focus on the external potential forces. Its particle model counterpart is studied in Ha-Shu (2018), mainly in one spatial dimension.

- Potential forces is possibly the easiest type of external force to study, since there still holds the energy dissipation.

Pairwise interaction: Carrillo-Choi-Tadmor-Tan (2016), Carrillo-Choi-Tse (2018), ...
Main results

• ‘Smooth solutions must flock’:
  
  • (1) Harmonic potential $\Psi(x) = \frac{a}{2} ||x||^2$, general $\phi$
  
  • (2) General convex potential, constant and large $\phi$

• Method: hypocoercivity (two different types)
Main results

• Existence of global smooth solutions: critical thresholds

  • (1) 1d: thresholds for global smooth solutions and blow-up
  • (2) 2d, harmonic potential (similar to He-Tadmor)
  • (3) 2d, general potential (including those without a flocking estimate!)

• All thresholds depend on the size of $\nabla^2 \Psi$: one reason why harmonic potential is special.

• Method: characteristics + spectral dynamics + $L^\infty$ estimates
Flocking: harmonic potential

Theorem 2.2. Let $\Psi(x) = \frac{a}{2} \|x\|^2$ be the harmonic potential. Let $(\rho, u)$ be a global smooth solution to (1.1). Assume $\rho_{\text{in}}$ has compact support. Then there holds the flocking estimate at exponential rate in both velocity and position:

$$E(t) := \int \int (\|u(x) - u(y)\|^2 + a\|x - y\|^2) \rho(x) \rho(y) \, dx \, dy \leq C E(0) e^{-\lambda t},$$

where $\lambda > 0$ depends on $a, \phi_-, \phi_+, m_0$, and $C > 0$ is an absolute constant.

- Velocity alignment happens together with spatial concentration! (because spatial deviation induces velocity deviation, if $\Psi$ is convex.)

- In the limit $a \to 0$, one has $\lambda = O(a)$. This means the strength of external potential may have big influence on flocking rate!
• Reduce to the case: mean velocity/position zero

• Energy estimate

\[ \partial_t \int \left( \frac{1}{2} \| \mathbf{u}(x, t) \|^2 + \frac{a}{2} \| x \|^2 \right) \rho(x, t) \, dx \leq -m_0 \phi - \int \| \mathbf{u} \|^2 \rho \, dx \]

• Cross term

\[ \partial_t \int \mathbf{u}(x, t) \cdot x \rho(x, t) \, dx \leq -\frac{a}{2} \int \| x \|^2 \rho \, dx + \left(1 + \frac{m_0^2 \phi^2}{a}\right) \int \| \mathbf{u} \|^2 \rho \, dx \]

• Use hypocoercivity to get exponential decay
Flocking:
general convex potential

**Theorem 2.3.** Let $\Psi(x)$ be strictly convex with bounded Hessian:

$$a\|y\|^2 \leq y^T \nabla^2 \Psi(x)y \leq A\|y\|^2, \quad \forall y \neq 0, \quad 0 < a < A \quad (2.4)$$

and $\phi$ is constant, and satisfies

$$m_0\phi > \frac{A}{\sqrt{a}} \quad (2.5)$$

Let $(\rho, u)$ be a global smooth solution to (1.1). Assume $\rho_{in}$ has compact support. Then there holds the flocking at exponential rate in both velocity and position:

$$\mathcal{E}(t) := \int \int (\|u(x) - u(y)\|^2 + a\|x - y\|^2)\rho(x)\rho(y) \, dx \, dy \leq C\mathcal{E}(0)e^{-\tilde{\lambda}t}, \quad (2.6)$$

where $\tilde{\lambda} > 0$ depends on $a, A, \phi_{-}, \phi_{+}, m_0$, and $C > 0$ depends on $a, A, m_0\phi$.

- Cannot reduce to ‘mean-zero’ case

- Total energy does NOT measure position concentration: cannot apply hypocoercivity directly
• A new Lyapunov functional

\[
F(t) = \frac{K}{2} \|x - y\|^2 + \langle x - y, u(x) - u(y) \rangle + \frac{\beta}{2} \|u(x) - u(y)\|^2
\]

\[
\|f(x, y), g(x, y)\| := \int \int f(x, y) \cdot g(x, y) \rho(x) \rho(y) \, dx \, dy, \quad \|f(x, y)\|^2 := \langle f(x, y), f(x, y) \rangle
\]

\[
\frac{dF}{dt} = \int \int [- (K \beta - 1) \|u(x) - u(y)\|^2 - (x - y) \cdot (\nabla \Psi(x) - \nabla \Psi(y))
- \beta (u(x) - u(y)) \cdot (\nabla \Psi(x) - \nabla \Psi(y))] \rho(x) \rho(y) \, dx \, dy
\]

• The second term is a good term if \( \Psi \) is convex.

• If \( K \) is large enough, then one can choose a proper \( \beta \) to absorb the bad term (the last term)
Regularity: 1d

**Theorem 2.7.** Let the space dimension $d = 1$. Assume $\Psi''$ is bounded below, and bounded above by:

$$\Psi''(x) \leq A, \quad \forall x \in \Omega$$

with $A$ being a constant satisfying

$$A < \frac{(m_0\phi_-)^2}{4}$$

Further assume that

$$\max_{x \in \text{supp } \rho_{in}} (\partial_x u_{in}(x) + (\phi \ast \rho_{in})(x)) > \frac{m_0\phi_-}{2} - \sqrt{\frac{(m_0\phi_-)^2}{4} - A}$$

then (1.1) admits global smooth solution.

- Positive $\Psi''$ induces blow-up; negative $\Psi''$ suppresses.
- The assumptions do NOT imply flocking.
Theorem 2.8. Assume
\[ \Psi''(x) \geq B, \quad \forall x \in \Omega \]

- If \( B \) is so large that
\[ B > \frac{(m_0\phi_+)^2}{4} \]
holds, then \( \partial_x u \) blows up to \(-\infty\) in finite time for any initial data.

- If (2.18) does not hold but \( B > 0 \), then \( \partial_x u \) blows up to \(-\infty\) in finite time if
\[ \partial_x u_{in}(x) + (\phi \ast \rho_{in})(x) < \frac{m_0\phi_+}{2} - \sqrt{\frac{(m_0\phi_+)^2}{4}} - B \]
for some \( x \). (notice that in this condition RHS \> 0)

- If \( B \leq 0 \), then \( \partial_x u \) blows up to \(-\infty\) in finite time if
\[ \partial_x u_{in}(x) + (\phi \ast \rho_{in})(x) < \frac{m_0\phi_-}{2} - \sqrt{\frac{(m_0\phi_-)^2}{4}} - B \]
for some \( x \). (notice that in this condition RHS \\leq 0)
Remark 3.2. One can obtain the explicit expression of $\mu_1$ from (3.35) by letting the good terms absorb the bad term exactly, i.e., solving the quadratic equation

$$\left(K_1 \mu_1 \right) \left( a \mu_1 \right) = A_2^2 \quad (3.44)$$

and get

$$\mu_1 = a K_2 A_2^2 r a^2 K_4 A_4 + 1 > 0 \quad (3.45)$$

Similar one obtains $\mu_2, \mu_3$ as

$$\mu_2, \mu_3 = \frac{1}{2} a a^2 K A_2^2 + K_2^2 \pm r \left( a^2 K A_2^2 + K_2^2 \right)^2 4 a \left( a K^2 A_2^2 + 1 \right) > 0 \quad (3.46)$$

Existence of global smooth solutions

4.1 1d case

The proof of the existence of global smooth solutions for 1d follows the technique of [1]: we analyze the ODE satisfied by the quantity $\partial_x u + \phi * \rho$ along characteristics.

Proof of Theorem 2.7.

Write $d := \partial_x u$. Differentiate the second equation of (4.1) with respect to $x$ to get

$$\partial_t \rho + u \partial_x \rho = \partial_t d + u \partial_x d + d^2 = Z \partial_x (x y) \rho (y) dy \quad (4.1)$$

Write $e = d + \phi * \rho \quad (4.2)$ and denote the time derivative along characteristics by '. Then

$$\rho' = -\rho (e - \phi * \rho) \quad \text{time derivative along characteristics}$$

$$e' = -e (e - \phi * \rho) - \Psi'' \quad \text{quadratic form in e}$$

If $e > 0$, then by (2.12),

$$e_0 = e (e - m_0) A = (e - m_0^2) + (m_0^2 - m_0^2) 4 A > 0 \quad (4.4)$$

Then by (2.13), one has

$$e_0 > 0, \quad \text{for } m_0^2 r (m_0^2) 4 A < e < m_0^2 + r (m_0^2) 4 A \quad (4.5)$$

By (2.14), initially $e > m_0^2 q (m_0^2) 4 A$ for all $x$. Therefore the same inequality persists for all time.

Also notice that $e_0 \leq e_2^2 a$ (4.6)

The 'magic quantity' $e = \partial_x u + \phi * \rho$

Carrillo-Choi-Tadmor-Tan, Shvydkoy-Tadmor
Regularity:
2d, harmonic potential

Theorem 2.9. Let the space dimension $d = 2$, and $\Psi(x) = \frac{a}{2}||x||^2$. There exists a positive constant $C_1$, depending on $m_0, \phi_-, \phi_+, a$ and $|\phi'|_\infty$ (the $L^\infty$ norm of $\phi'$), such that the following holds: Assume

$$c_1^2 := m_0^2 \phi_-^2 - \left( \max_{x \in \text{supp } \rho_{in}} |(\eta_S)_{in}(x)| + C_1 E_\infty(0) \right)^2 - 4a > 0 \quad (2.21)$$

where $\eta_S$ is the difference between the two eigenvalues of the symmetric matrix $(\nabla u + (\nabla u)^T)/2$. Assume

$$\max_{x \in \text{supp } \rho_{in}} (\nabla \cdot u_{in}(x) + (\phi \ast \rho_{in})(x)) \geq 0 \quad (2.22)$$

then (1.1) admits global smooth solution.

- The estimate is good only when the size of $a$ is moderate: large $a$ will blow up the $4a$ term (the same reason as 1d), and small $a$ will blow up the $C_{-1}$ term (because flocking estimate is bad).
\[ \partial_t M + u \cdot \nabla M + M^2 = -(\phi \ast \rho)M + R - \nabla^2 \Psi \quad M_{ij} = \partial_j u_i \]

\[ R_{ij} = \partial_j \phi \ast (\rho u_i) - u_i (\partial_j \phi \ast \rho) \]

- **Spectral dynamics**

\[ e' = \frac{1}{2} \left( 4\omega^2 + (\phi \ast \rho)^2 - \eta_S^2 - e^2 - 2\Delta \Psi \right) \quad e = \nabla \cdot u + \phi \ast \rho \quad \omega = (\partial_1 u_2 - \partial_2 u_1)/2 \]

\[ \eta' + e \eta = q := \langle s_2, R_{sym}s_2 \rangle - \langle s_1, R_{sym}s_1 \rangle - \langle s_2, \nabla^2 \Psi s_2 \rangle + \langle s_1, \nabla^2 \Psi s_1 \rangle \]

\[ s_i : \text{(orthonormal) eigenvectors of S, the symmetric part of M} \]

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H. Liu-Tadmor (2002), He-Tadmor
For harmonic potential,

\[ \eta_S' + e \eta_S = q := \langle s_2, R_{sym} s_2 \rangle - \langle s_1, R_{sym} s_1 \rangle - \langle s_2, \nabla^2 \Psi s_2 \rangle + \langle s_1, \nabla^2 \Psi s_1 \rangle \]

Flocking estimate \(\rightarrow\) exponential decay of \(R\)

If \(e \geq 0\), then

\[ |\eta_S(t)| \leq |(\eta_S)_{in}| + \int_0^\infty |R(s)| ds, \quad \forall t \]

Use this to propagate \(e \geq 0\)

---

Similar to He-Tadmor
Regularity: 2d, general potential

**Theorem 2.10.** Let the space dimension \(d = 2\). Assume \(\nabla^2 \Psi\) is bounded:

\[
\|\nabla^2 \Psi(x)\| \leq A
\]

Assume that there is an a priori estimate

\[
\max_{t \geq 0, x \in \text{supp}} \|u(x,t)\| \leq u_{max}
\]

for some constant \(u_{max}\). If there hold

\[
C_2 := 8|\phi'|_\infty m_0 u_{max} + 2A < m_0^2 \phi_-^2 / 2 - 2A
\]

and

\[
\max_{x \in \text{supp} \rho_{in}} |(\eta_s)_{in}(x)| \leq \sqrt{(m_0^2 \phi_-^2 / 2 - 2A) + (m_0^2 \phi_-^2 / 2 - 2A)^2} - C_2^2
\]

and

\[
\max_{x \in \text{supp} \rho_{in}} (\nabla \cdot u_{in}(x) + (\phi \ast \rho_{in})(x)) > \sqrt{(m_0^2 \phi_-^2 / 2 - 2A) - (m_0^2 \phi_-^2 / 2 - 2A)^2} - C_2^2
\]

then (1.1) admits global smooth solution.
• For general potential,

\[ \eta_S' + e\eta_S = q := \langle s_2, R_{\text{sym}}s_2 \rangle - \langle s_1, R_{\text{sym}}s_1 \rangle - \langle s_2, \nabla^2\Psi s_2 \rangle + \langle s_1, \nabla^2\Psi s_1 \rangle \]

• No flocking estimate: R does not decay. The best we can do is a uniform-in-time bound: the \( u_{\{\text{max}\}} \) assumption

• \( \nabla^2\Psi \) is not identity matrix \( \rightarrow \) last two terms are \( O(1) \)

• **New idea:** make use of the good term \( e\eta_S \)

• Need to propagate a **positive** lower bound of \( e \)
\textbf{L^\infty estimate}

This allows non-convex potentials (even with multiple local minima)

**Proposition 2.11.** Assume that there exists constant $A, a > 0, X_0 > 0$ such that

\[
\frac{a}{2} \|x\|^2 \leq \Psi(x) \leq \frac{A}{2} \|x\|^2, \quad a \|x\| \leq \|\nabla \Psi(x)\| \leq A \|x\|, \quad \forall x \in \Omega, \|x\| \geq X_0
\]  

(2.29)

Then there exists a constant $u_{\text{max}}$, depending on $a, A, \phi_-, \phi_+, m_0, E(0)$, where $E(t)$ is the total energy

\[
E(t) = \int \left( \frac{1}{2} \|u(x, t)\|^2 + \Psi(x) \right) \rho(x, t) \, dx
\]  

(2.30)

such that

\[
\max_{t \geq 0, x \in \text{supp } \rho_{\text{in}}} \|u(x, t)\| \leq u_{\text{max}}
\]  

(2.31)

- Method: hypocoercivity along characteristics

\[
F(x, t) = \frac{1}{2} \|u(x, t)\|^2 + \Psi(x) + cu(x, t) \cdot \nabla \Psi(x)
\]
Conclusion

• ‘Smooth solutions must flock’:
  
  • (1) Harmonic potential $\Psi(x) = \frac{a}{2} \|x\|^2$, general $\phi$

  • (2) General convex potential, constant and large $\phi$

• Existence of global smooth solutions: critical thresholds

  • (1) 1d: thresholds for global smooth solutions and blow-up

  • (2) 2d, harmonic potential (similar to He-Tadmor)

  • (3) 2d, general potential (including those without a flocking estimate!)