Complex oscillator networks and applications in power systems

ETH-ITS lecture series “Collective dynamics, control and imaging”

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A brief history of sync

Christiaan Huygens (1629 – 1695)
- physicist & mathematician
- engineer & horologist
observed “an odd kind of sympathy”
[Letter to Royal Society of London, 1665]

Recent reviews, experiments, & analysis
[M. Bennet et al. ’02, M. Kapitaniak et al. ’12]

Acknowledgements

Dominic Groß
Marcello Colombino

A field was born

- sync in mathematical biology [A. Winfree ’80, S.H. Strogatz ’03, ...]
- sync in physics and chemistry [Y. Kuramoto ’83, M. Mézard et al. ’87...]
- sync in neural networks [F.C. Hoppensteadt and E.M. Izhikevich ’00, ...]
- sync in complex networks [C.W. Wu ’07, S. Boccaletti ’08, ...]
- ... and numerous technological applications (reviewed later)
Coupled phase oscillators

- Various models of oscillators & interactions
- Canonical coupled phase oscillator model: [A. Winfree ’67, Y. Kuramoto ’75]

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$

- $n$ oscillators with phase $\theta_i \in S^1$
- Non-identical natural frequencies $\omega_i \in \mathbb{R}^1$
- Elastic coupling with strength $a_{ij} = a_{ji}$
- Undirected & connected graph $G = (V, E, A)$

Phenomenology & challenges in synchronization

Transition to synchronization is a trade-off: coupling vs. heterogeneity

- Strong coupling & homogeneous
- Weak coupling & heterogeneous

"Surprisingly enough, this seemingly obvious fact seems difficult to prove."
(Y. Kuramoto’s conclusion after proposing the model)

Two open central questions:
- Quantify “coupling” vs. “heterogeneity”
- Basin of attraction for synchronization

Outline

- Introduction
- From Coupled Oscillators to Inverter-Based Power Systems
- Coupled Phase Oscillators and Inverter Droop Control
- Consensus-Inspired Approach to Synchronization
- Conclusions
Renewable/distributed/inverter-based generation on the rise

From coupled oscillators to AC power systems

(simplified) swing equation model of interconnected synchronous generators:

\[ M_i \ddot{\theta}_i + D_i \dot{\theta}_i = p_i^* - \sum_j y_{ij} \sin(\theta_i - \theta_j) \]

where \( M_i, D_i, p_i^* \) are inertia, damping, power injection set-point of generator \( i \)

New primary sources

focus today on inverter-based generation

Modeling: signal space in three-phase AC power systems

\[
\begin{bmatrix}
    x_a(t) \\
    x_b(t) \\
    x_c(t)
\end{bmatrix}
= \begin{bmatrix}
    x_a(t + T) \\
    x_b(t + T) \\
    x_c(t + T)
\end{bmatrix}
\]

balanced (nearly true)

\[
\begin{bmatrix}
    \sin(\delta(t)) \\
    \sin(\delta(t) - \frac{2\pi}{3}) \\
    \sin(\delta(t) + \frac{2\pi}{3})
\end{bmatrix}
\]

periodic with 0 average

\[
\frac{1}{T} \int_0^T x_i(t) dt = 0
\]

so that

\[
x_a(t) + x_b(t) + x_c(t) = 0
\]

synchronous (desired)

\[
\begin{bmatrix}
    \sin(\delta_0 + \omega_0 t) \\
    \sin(\delta_0 + \omega_0 t - \frac{2\pi}{3}) \\
    \sin(\delta_0 + \omega_0 t + \frac{2\pi}{3})
\end{bmatrix}
\]

const. freq & amp

⇒ const. in rot. frame

assumption: signals are balanced ⇒ 2d-coordinates \( x(t) = [x_a(t), x_b(t)] \)
(equivalent representation: complex-valued polar/phasor coordinates)
Modeling: the inverter

- **terminal signals**: voltage $v_k \in \mathbb{R}^2$ and output current $i_{o,k} \in \mathbb{R}^2$
- **controllable signal**: switching modulation signal $m_k$

$C_k \frac{d}{dt} v_k = -G_k v_k - i_{o,k} + i_k$
$\sim$ controllable voltage source

$\Rightarrow$ common abstraction:
- **direct control** of current $i_k$:

Modeling: the network equations

- **branch dynamics**: each branch is series of resistance $r_{ij}$ & inductance $L_{ij}$
- **time-scale separation**: all network signals are assumed to be in synchronous steady-state $\dot{x}(t) = \omega_0 J x(t)$ where $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \sim \sqrt{-1}$
- **admittance matrix** $Y = Y^T$ with admittances $y_{ij} = (r_{ij} + \omega_0 L_{ij})^{-1}$

$\Rightarrow$ balance equations: $i_o = Y v$ with terminal currents/voltages $(i_o, v)$

Objectives for decentralized control design

We aim to stabilize a target trajectory $(v(t), i_o(t))$ satisfying the following:

- **frequency stability** at synchronous frequency $\omega_0$:
  \[
  \frac{d}{dt} v_k(t) = \omega_0 J v_k(t)
  \]
  $\sim$ synchronization to desired harmonic waveform

- **voltage regulation** to desired voltage magnitudes $v_k^*$:
  \[
  \|v_k(t)\| = v_k^*
  \]
  $\sim$ stabilization of desired amplitudes (assume wlog that $v_k^* = v^*$)

- **power injection set-points** for active & reactive power $\{p_k^*, q_k^*\}$:
  \[
  v_k^T i_{o,k} = p_k^* \quad , \quad v_k^T J i_{o,k} = q_k^*
  \]
  $\sim$ stabilization of desired angle set-points $\{\theta_{kj}\}$

Overview of oscillator-based control strategies for inverters

- classic droop control (brief review)
- consensus-inspired approach (today)
- synchronous generator emulation
- virtual van-der-Pol oscillator control

[Zhong, Weiss, '11, D’Arco, Suul '13, Bevrani, Sie, Miura '14, Jouini, Arghir, FD '16]
[Johnson, Dhople, Hamadheh, & Krein, '13, Sinha, Johnson, Ainsworth, FD, Dhople '15]
Droop control of power inverters

**key idea**: replicate the generator swing equation for the inverter:

\[
M_i \ddot{\theta}_i + D_i \dot{\theta}_i = p_i^* - \sum_j y_{ij} \sin(\theta_i - \theta_j)
\]

**Standard implementation**:

- **measure** terminal currents and voltages \((v(t), i_o(t))\)
- **process** measurements:
  - active power \(p_k = v_k^\top i_{o,k}\)
  - reactive power \(q_k = v_k^\top J i_{o,k}\)
- **proportional control** of terminal voltage waveform:
  \[
  \frac{d}{dt} \left[ \frac{\theta_k}{\|v_k\|} \right] = (v_k^*, p_k^*, q_k^*) - (\|v_k\|, p_k, q_k)
  \]

Closer look at droop control for a lossless network

**key idea**: replicate the generator swing equation for the inverter:

\[
M_i \ddot{\theta}_i + D_i \dot{\theta}_i = p_i^* - \sum_j y_{ij} \sin(\theta_i - \theta_j)
\]

- \(p - \omega\) droop mimicking a generator:
  \[
  \frac{d}{dt} \theta_k = \omega_0 + k (p_k^* - p_k) = \omega_0 + kp_k^* - k \cdot \sum_j y_{kj} \|v_j\| \|v_k\| \sin(\theta_k - \theta_j)
  \]

- analogous \(q - \|v\|\) droop (many variations):
  \[
  \frac{d}{dt} \|v_k\| = k (q_k^* - q_k) + k (\|v^* - \|v_k\|\| \cdot \|v_k\|
  \]

**Standard analysis** of lossless droop/generators/oscillators

**energy function**: inductive energy + quadratic amplitude error

\[
V(v) = \sum_{j<k} y_{jk} (\|v_k\|^2 - \|v_k\|\|v_j\| \cos(\theta_k - \theta_j)) + \frac{1}{2} \sum_k (v^* - \|v_k\|)^2
\]

- \(p - \omega\) droop:
  \[
  \frac{d}{dt} (\theta_k - \omega_0 t) = kp_k^* - k \cdot \frac{\partial V}{\partial \theta}
  \]

- \(q - \|v\|\) droop:
  \[
  \frac{d}{dt} \|v_k\| = kq_k^* - k \cdot \|v_k\| \cdot \frac{\partial V}{\partial \|v_k\|}
  \]

⇒ closed loop is weighted gradient flow if all \(p_k^*\) are identical and \(q_k^* = 0\)

**standard analysis**: if equilibria exists, if \((p_k^*, q_k^*)\) are sufficiently small, if lossless, if \(\nabla^2 V(v^*)\) is positive definite, \(\ldots \implies\) local asymptotic stability

analysis has many limitations & droop has similar practical limitations!
Recall: objectives for decentralized control design

We aim to stabilize a target trajectory \((v(t), i_o(t))\) satisfying the following:

1. **frequency stability** at synchronous frequency \(\omega_0\):
   \[
   \frac{d}{dt} v_k(t) = \omega_0 J v_k(t)
   \]
   \(\sim\) synchronization to desired harmonic waveform

2. **voltage regulation** to desired voltage magnitudes \(v^*_k\):
   \[
   \|v_k(t)\| = v^*
   \]
   \(\sim\) stabilization of desired amplitudes

3. **power injection set-points** for active & reactive power \(\{p^*_k, q^*_k\}\):
   \[
   v_k^T i_o,k = p^*_k, \quad v_k^T J i_o,k = q^*_k
   \]
   \(\sim\) stabilization of desired angle set-points \(\{\theta^*_k\}\)

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Overview of design strategy

**Step 1:** construct target dynamics \(f^*(v(t))\) for the terminal voltages so that \(\frac{dv}{dt} = f^*(v(t))\) is stable and satisfies the three control objectives.

**Step 2:** achieve desired target dynamics \(\frac{dv}{dt} = f^*(v(t))\) via fully decentralized current controllers \(i^*_k(v_k, i_{k,o})\).

- **local measurements** \((v_k, i_{k,o})\)
- **local set-points** \((v^*_k, p^*_k, q^*_k)\)
- **unknown:** global set-points for angles \(\{\theta^*_k\}\) and nonlocal measurements of \((v_j, i_{o,j})\)

**Step 3:** implement control via switching modulation signal (not today).

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Step 1: desirable closed-loop target dynamics

Objectives: frequency, phase, and voltage stability

\[
\frac{d}{dt} v = \omega_0 J v + \eta \cdot e_{\|v\|}(v) + \alpha \cdot e_{\|v\|}(v)
\]

\(\text{rotation at } \omega = \omega_0\) \quad \text{phase error} \quad \text{magnitude error}

(i) **synchronous rotation:**
\[
\frac{d}{dt} v = \omega_0 J v = \omega_0 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} v
\]

(ii) **phase stabilization:**
\[
\frac{d}{dt} v = e_{\|v\|}(v) = \sum_{j=1}^n w_{jk}(v_j - R(\theta^*_{jk})v_k)
\]

(iii) **magnitude stabilization:**
\[
\frac{d}{dt} v = e_{\|v\|}(v_k) = (v^* - \|v_k\|)v_k
\]

Gains: \(\alpha_k, \eta_k > 0\) for each \(k\) and \(w_{jk} \geq 0\) for relative phases amongst \(\{j, k\}\)
Illustration of desirable closed-loop dynamics

\[ \frac{d}{dt} \mathbf{v}_k = \omega_0 \mathcal{J} \mathbf{v}_k + \eta \cdot \mathbf{e}_\theta(\mathbf{v}) + \alpha \cdot \mathbf{e}_{||\mathbf{v}||}(\mathbf{v}) \]

Decentralized implementation of phase error dynamics

\[ \mathbf{e}_{\theta,k}(\mathbf{v}) = \sum_j w_{jk}(\mathbf{v}_j - R(\theta_{jk}^*) \mathbf{v}_k) = \sum_j w_{jk}(\mathbf{v}_j - \mathbf{v}_k) + \sum_j w_{jk}(I - R(\theta_{jk}^*)) \mathbf{v}_k \]

Phase error:

\[ e_{\theta,1}(\mathbf{v}) = v_2 - R(\theta_{21}^*) v_1 \]
\[ e_{\theta,2}(\mathbf{v}) = v_1 - R(\theta_{12}^*) v_2 \]

Magnitude error:

\[ e_{||\mathbf{v}||,1}(\mathbf{v}) = (v^* - ||\mathbf{v}_1||) v_1 \]
\[ e_{||\mathbf{v}||,2}(\mathbf{v}) = (v^* - ||\mathbf{v}_2||) v_2 \]

insight I: non-local measurements from communication through physics:
assume that all lines are homogeneous \( r_{ij} / \ell_{ij} = \kappa = \text{const.} \), then

\[ R(\kappa) i_o = R(\kappa) \mathcal{J} \mathbf{v} = \mathcal{L} \mathbf{v} \]

local feedback Laplacian matrix Laplacian feedback

insight II: angle set-points & line-parameters from power flow equations:

\[ p_k^* = v^* \sum_j \left( \frac{R_j (1 - \cos(\theta_j^*)) - \omega_0 L_j \sin(\theta_j^*)}{R_j^2 + \omega_0^2 L_j^2} \right) \]
\[ q_k^* = v^* \sum_j \left( \frac{\omega_0 L_j (1 - \cos(\theta_j^*)) + \gamma \omega_0 L_j \sin(\theta_j^*)}{R_j^2 + \omega_0^2 L_j^2} \right) \]

\[ \Rightarrow K_k(\theta_j^*) = \frac{R(\kappa)}{v^*} \begin{bmatrix} p_k^* - q_k^* \\ -q_k^* \end{bmatrix} \]

global parameters local parameters

Surprising (?) connections revealed in polar coordinates

Closed loop:

\[ \frac{d}{dt} \mathbf{v} = \omega_0 \mathcal{J} \mathbf{v} + \eta \cdot \mathbf{e}_\theta(\mathbf{v}) + \alpha \cdot \mathbf{e}_{||\mathbf{v}||}(\mathbf{v}) \]

rotation at \( \omega = \omega_0 \) phase error magnitude error

Polar coordinates in lossless case \( r_{ij} = 0 \) & near nominal voltage \( ||\mathbf{v}_k|| \approx 1 \):

\[ \frac{d}{dt} ||\mathbf{v}_k|| = \eta \left( \frac{q_k^*}{v^*} - \frac{q_k}{||\mathbf{v}_k||^2} \right) ||\mathbf{v}_k|| + \frac{\alpha}{v^*} (v^* - ||\mathbf{v}_k||) ||\mathbf{v}_k|| \]

magnitude

\[ \approx \eta (q_k^* - q_k) ||\mathbf{v}_k|| + \alpha (v^* - ||\mathbf{v}_k||) ||\mathbf{v}_k|| \]

\[ q \approx ||\mathbf{v}|| \text{ droop} \]

\[ \frac{d}{dt} \theta_k = \omega_0 + \eta \left( \frac{p_k^*}{v^*} - \frac{p_k}{||\mathbf{v}_k||^2} \right) \]

\[ \Rightarrow \omega_0 + \eta (p_k^* - p_k) \quad p - \omega \text{ droop} \]

Kuramoto oscillator

Step 2: decentralized implementation of target dynamics

open-loop system: controllable voltage sources + network coupling

\[ C_k \frac{d}{dt} \mathbf{v}_k = -G_k \mathbf{v}_k - i_o,k + i_k, \quad k \in \{1, \ldots, n\} \]
i_o = \gamma \mathbf{v} \quad \text{inverter dynamics}

network interconnection

Target dynamics:

\[ \frac{d}{dt} \mathbf{v} = \omega_0 \mathcal{J} \mathbf{v} + \eta \cdot \mathbf{e}_\theta(\mathbf{v}) + \alpha \cdot \mathbf{e}_{||\mathbf{v}||}(\mathbf{v}) \]

rotation at \( \omega = \omega_0 \) phase error magnitude error

\[ \mathbf{v} \text{ known: local measurements } (v_k, i_{o,k}) \& set-points } (v_k^*, p_k^*, q_k^*) \]

\[ \mathbf{v} \text{ unknown: global set-points for } \theta_j \text{ and nonlocal measurements of } (v_j, i_{o,j}) \]
Closed-loop stability analysis

Closed loop: \[ \frac{dv}{dt} = w_0 + \eta \cdot (-L + K) \cdot v + \alpha \cdot \text{diag}(v^* - \|v_k\|) \cdot v \]

in rotating frame

Stationary target sets in rotating coordinate frame:

- \( S = \{ v \in \mathbb{R}^n \mid v_k = R(\theta_{kj}^*) v_1 \} \) set of correct relative angles
- \( A = \{ v \in \mathbb{R}^n \mid \|v_k\| = v^* \} \) set of correct magnitudes

Main result: \( \mathcal{T} = S \cap A \) is almost globally asymptotically stable if the grid parameters, angle set-points, and control gains satisfy a mild condition.

Discussion of stability condition

Energy-like function centered at \( \{ \theta_{kj}^* \} \) and with free-floating amplitudes:

\[ V(v) = \sum_{j<k} w_{jk} (\|v_k\|^2 - \|v_k\|\|v_j\| \cos(\theta_k - \theta_j - \theta_{kj}^*)) = v^T P v \]

\( \Rightarrow \) closed loop is gradient flow \( \dot{v} = -\nabla V(v) \) for \( \alpha = 0 \) and \( \{ \theta_{kj}^* \} = \emptyset \)

Condition \( \Leftrightarrow \) energy function non-increasing despite \( \alpha > 0 \) & \( \{ \theta_{kj}^* \} \neq \emptyset \):

\[ \eta \left( (K - L)^T P + P(K - L) \right) + 2 \alpha P \leq 0 \]

Remark: the assumption can always be met in a connected grid if

- slow/fast loops: amplitude gain \( \alpha < \) synchronizing gain \( \eta \)
- not too heavy loading: sufficiently small relative angle set-points \( \{ \theta_{kj}^* \} \)

\( \Rightarrow \) stability condition is very reasonable (always met) for practical scenarios

Almost global synchronization

to trajectory with prescribed frequency, voltage amplitudes, & active/reactive power injections

Exponential stability of \( S \):

under our assumption, \( \|v\|^2_S \) is a Lyapunov function for \( S \).

Stability of \( A \) relative to \( S \):

provided \( v(0) \in S \setminus \{0\} \), the set \( A \) is asymptotically stable.

\( Z \) has measure zero:

the region of attraction of \( \{0\} \), termed \( Z \), has measure zero.

Continuity argument:

almost all trajectories (not in \( Z \)) approaching \( S \) must (by continuity) converge to \( S \cap A \).

Simulation example

Voltage time series

Voltage phase space
Conclusions

Summary:
- oscillator networks & power applications
- droop control & Kuramoto oscillators
- design of decentralized oscillator control

Outlook:
- robustness & performance analysis
- fair comparison of different approaches
- experimental validation

Insights for coupled oscillators:
- Kuramoto models rarely (almost never) admit almost global synchronization
- it pays off to work in $\mathbb{R}^2$ rather than $S^1$