Two Classical models of Quantum Dynamics

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Outline:

• Review of dynamics and localization for Random Schrödinger H on $\ell_2(\mathbb{Z}^d)$

$$H = -\Delta + \lambda V(j); \quad j \in \mathbb{Z}^d, \quad V(j) \; iid.$$

• Motion of Classical particle on Manhattan lattice deflected by random obstructions. Quantum Network Model

• Edge Reinforced Random Walk - P. Diaconis
SUSY Statistical mechanics M. Zirnbauer

• Classical dynamics: Kesten, Papanicolaou, Komorowski and Ryzhik

$$H = \dot{x}^2 + \lambda V(x), \quad x \in \mathbb{R}^d$$

Localization and Dynamics for Random Schrödinger

$$H = -\Delta + \lambda V(j); \quad j \in \mathbb{Z}^d, \quad V(j) \ iid.$$

$$\mathbb{E}(V(j)) = 0; \quad \mathbb{E}(V(j)^2) = 1$$

$$-\Delta e^{i\, j\cdot p} = \sum_{\alpha=1}^{d} (2 - 2cos\, p_\alpha) e^{i\, j\cdot p} \approx p^2\, e^{i\, j\cdot p}$$

$$i \frac{\partial}{\partial t}\, \psi = H\, \psi; \quad \psi = \psi(t,j); \quad \psi(t=0,j) = \delta_0(j)$$

$$R^2(t) \equiv \mathbb{E} \sum_j |\psi(t,j)|^2 |j|^2\, ; \quad \sum_j |\psi(t,j)|^2 = 1;$$

$$R^2(t) \approx Ct^2\, ; \quad \textbf{Ballistic}\, ; \quad if\ \ \lambda = 0\ or\ V\ \ periodic$$

Localization

$$R^2(t) \leq Const;$$ Uniform Localization

All eigenfunctions of H decay rapidly about some point $c(\alpha) \in \mathbb{Z}^d$:

$$|\psi_\alpha(j)| \leq C\, e^{-|j-c(\alpha)|/\ell(E,\lambda)}; \quad \ell(E,\lambda) = \text{localization length}$$

Theorem In one dimension, $\mathbb{Z}^1$, all eigenstates are uniformly localized $\ell(E,\lambda) \approx \lambda^{-2}$, *all* $\lambda > 0$

Furstenberg, Goldsheid, Molchanov, Pastur, Ledrappier, Margulis

Conjecture A : In $\mathbb{Z}^2$, Localization for all $\lambda > 0$; $\quad \ell(E,\lambda) \leq e^{\lambda^{-2}}$

Conjecture A' : In a 1D strip of width W, $\quad \ell(E,\lambda) \leq W\lambda^{-2}$

Localization on $\mathbb{Z}^d$ for $d \geq 2$

If $\lambda \gg 1$ then

1) All eigenstates of H are exponentially localized.

2) $R(t)^2 \leq Const$ for all d. Note this is false in the continuum!

If $|\lambda| \ll 1$, there is localization for $E \leq -\lambda^2 c_d$

In 3D **expect** extended states for E above $-\lambda^2 c'_d$ - Anderson transition.

Anderson, Thouless, Fröhlich, Sp, Martinelli, Scoppola, Simon, Wolff, Aizenman, Molchanov ...

Localization via Green's function

$$G_{E+i\epsilon}(x,y) \equiv [H - E - i\epsilon]^{-1}(x,y), \quad x,y \in \mathbb{Z}^d, \quad \epsilon > 0$$

Fractional moment method:

$$\mathbb{E}\,|G_{E+i\,0}(x,y)|^{1/2} \le e^{-|x-y|/\ell(E)} \Rightarrow \text{Localization near E}$$

Aizenman, Molchanov, Hundertmark, Friedrich, Schenker.

Conjecture - 3D quantum diffusion:

$$\sum_{x} e^{ix\cdot p} \, \mathbb{E}|G_{E+i\epsilon}(x,0)|^2 \approx C[D(E,\epsilon)p^2 + \epsilon]^{-1}$$

Diffusion constant $= D(E,\epsilon) = D(E)$; $\epsilon \downarrow 0$, time scale $= \epsilon^{-1}$

$$\mathbb{E}|G_{E+i\epsilon}(x,x)|^2 \leq Const; \Rightarrow \text{Absolutely Continuous spectrum}$$

Quantum Dynamics in 3D

$$H = -\Delta + \lambda V(j); \quad j \in \mathbb{Z}^d, \quad V(j) \; iid.$$

Conjecture - 3D quantum diffusion:

$$R^2(t) \approx Dt, \quad \textit{for small } \lambda$$

The wave spread ballistically until time scale λ^{-2}. After this time we expect quantum diffusion.

Erdős-Salmhofer-Yau get control for times $\leq \lambda^{-2-\delta}$, δ *small*.

Manhattan Pinball

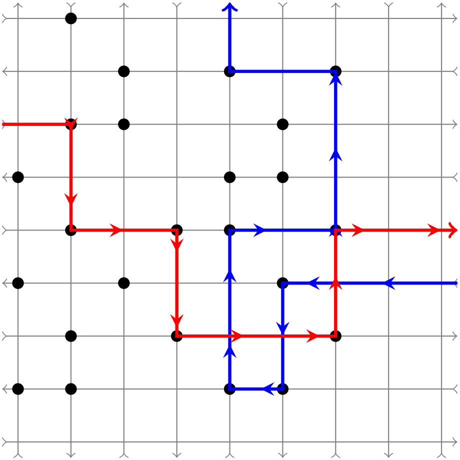
Quantum Network Model with random scatterers

Motivation: Chalker's network model Integer Quantum Hall.

Particle moves on $\mathbb{Z}^2$ with alternating orientations for the streets.

Independently with Prob $0 < p < 1$, an obstruction is randomly placed at a vertex. Particle moves in the direction of orientation until it meets an obstruction. Then it turns in the direction of orientation.

(Beamond, Cardy, Owczarek; Gruzberg, Ludwig, Read)



● obstruction

Figure: Manhattan Lattice

Theorem If $p > 1/2$ then all orbits are closed with probability 1

Localization. Proof by percolation.

Conjecture: All loops are closed for any $p > 0$

Average loop length $\ell \approx e^{c\,p^{-2}} \gg 1.$

At intermediate distances paths should look like random walk: diffusion at scales $p^{-m}, m \geq 2$.

Problems: Prove localization for some $p < 1/2$

Show that on a 1D cylinder of width W, $\ell(W) \leq Const\ p^{-2} W$.

Note easy to prove: $\ell(W) \leq e^{C\,W}$

Linearly Edge Reinforced Random Walk

History dependent walk $W_n \in \mathbb{Z}^d$, $n \in \mathbb{Z}^+$:

Walk takes nearest neighbor steps and favors edges $j, k, \in \mathbb{Z}^d$, $|j-k| = 1$, it has visited in the past.

Introduced by P. Diaconis in 1986 while wandering the streets of Paris. He liked to return to streets he had visited in the past.

Why Linearly Edge Reinforced?

Related to Polya's Urn.

Partially exchangeable process - generalization of de Finetti

Not Markovian but is a superposition of Markov Processes

Equivalent to a random walk in a random environment - must average over environment: $\mathcal{E}$

Definition of Reinforced Random Walk

Let $C_{jk}(n)$ = number of times the walk has crossed edge jk up to time n and let $\beta > 0$.

$$Prob\{W_{n+1} = k | W_n = j\} = \frac{1 + C_{jk}(n)/\beta}{\mathcal{N}_\beta}, \quad |j - k| = 1.$$

$\mathcal{N}$ is the normalization: $\mathcal{N}_\beta = \sum_{k'}(1 + C_{jk'}(n)/\beta), \quad |j - k'| = 1$

$0 < \beta \ll 1$, **strong** reinforcement, (high temperature)
$\beta \gg 1$, **weak** reinforcement, (low temperature)

Long time behavior of W_n , n, large?

Is ERRW recurrent? Localized?

Localization:

$$Prob_{\beta}\{|W_n - W_0| \geq R\} \leq Ce^{-R/\ell}\ , \quad \ell(\beta) = \text{localization length}$$

Is ERRW Transient? Diffusive?

Is there a Phase Transition as we vary the reinforcement β ?

P. Diaconis and D. Coppersmith (1986):

ERRW $\approx$ random walk in a random environment.

Environment: The rate at which an edge j, j' is crossed $w_{j,j'} > 0$ are correlated random variables. (conductances)

Explicit Joint Distribution of $w_{j,j'} > 0$ first appeared in an unpublished paper Diaconis and Coppersmith. It given by **statistical mechanics**.

The generator for the RW is a weighted Laplacian L

$$v^t \cdot L\, v = \sum_{|j-j'|=1} w_{j,j'} (v_j - v_{j'})^2$$

However, L is NOT uniformly elliptic.

Important: The distribution of $w_{j,j'}$ depends on the **starting point** of the Walk.

Relation to Random Schrödinger

Spectral properties of Random Schrödinger Equivalent **dual model** in statistical mechanics with Supersymmetric Hyperbolic symmetry - 1982 Efetov: $U(1,1|2)$ SUSY - Rigorous equivalence but **very** complicated.

In 1991 Martin Zirnbauer defined a simplified version of Efetov's dual model: SUSY Hyperbolic sigma model $H^{2|2}$

In any dimension, spin correlations of $H^{2|2}$ can be expressed as a random walk in a random environment.

$$w_{j,j'} = e^{t_j + t_{j'}}, \quad \text{joint distribution} \ \equiv e^{-E_{SUSY}(\beta,\{t_j\})}.$$

The expectation is denoted by $\mathcal{E}$.

The SUSY Hyperbolic Model - $H^{(2|2)}$

$$E_{SUSY}(\{t_j\}) =$$

$$\beta \sum_{j\sim j'} cosh(t_j - t_{j'}) - 1/2 \log \det L_{\beta,\epsilon}(t) + \epsilon \sum cosh\, t_j$$

where L is weighted Laplacian:

$$[v\,; L_{\beta,\epsilon}(t)\, v] = \beta \sum\nolimits_{(j'\sim j)} \mathrm{e}^{t_j+t_{j'}} (v_j - v_{j'})^2 + \epsilon \sum\nolimits_{k\in\Lambda} \mathrm{e}^{t_k} v_k^2$$

Spin-Spin correlation:

$$< e^{t_0+t_x} L_{\beta,\epsilon}(t)^{-1}(0,x) >_{SUSY} (\beta,\epsilon)$$

Some results in 1 and 2D

R. Pemantle analyzed ERRW on the Regular tree. Showed that it has sharp transition in β from recurrent to transient.

Merkl and Rolles studied one dimensional strips of Width W and show the ERRW is localized with $\ell(W) \leq \beta W$

In 2D Merkl and Rolles prove the conductance $\mathcal{E}\, w_{jj'}^{1/4}$ has a power law decay away from the origin - via a deformation argument.

Sabot and Zeng: In 2D ERRW is recurrent for all β.

Conjecture: In 2D the walk is exponentially localized for all β.

Theorem (Disertori-S-Zirnbauer '10) **Phase transition**:

For $\beta \gg 1$ and $d \geq 3$ the conductances $w_{jj'}$ in the $H^{2|2}$ model are bounded above and below with high probability - **Transient, quasi-diffusion**

For $0 < \beta \ll 1$, Conductance goes 0, **Recurrent, Localization**

Kozma and later Sznitman pointed out *similarities* of $H^{2|2}$ to the formulas for **ERRW**.

Sabot and Tarres ('12) Showed how to modify $H^{2|2}$ model to get the law for **ERRW**.

Phase Transition for **ERRW** in 3D

Theorem (Sabot-Tarres, Angel-Crawford-Kozma) For **strong** reinforcement, $0 < \beta \ll 1$,
ERRW is recurrent and we have Localization:

$$Prob\{|W(t) - W(0)| \geq R\} \leq Ce^{-R/\ell(\beta)}, \quad \mathcal{E}\, W^2(t) \leq Const$$

and $\ell(\beta)$ is the localization length.

Theorem (Disertori-Sabot-Tarres) For **weak** reinforcement, $\beta \gg 1$, and $d \geq 3$, $W(t)$ is Transient, quasi-diffusion.

Outlook and Problems

A) For Random Schrödinger show that $R^2(t) \leq t^{2-\delta}$
strictly sub-ballistic.

B) For Manhattan model show that for long time scales $t \leq p^{-m}$ typical trajectories behave like a **random walk** for p is small. (For ESY, m $= 1.1$)

C) Is there a sharp transition in 3D?: Localization for $0 < \beta < \beta_c$ and Diffusion for $\beta > \beta_c$

D)Multi-fractal transition is expected in 3D at β_c for both Random Schrödinger and ERRW. Is MF present on Bethe lattice for ERRW?

E) In 2D, is ERRW is localized for weak reinforcement? For all $\beta > 0$, $\ell(\beta) \approx e^{C\beta}$, $\mathcal{E} w_{j,j'}^{1/4} \approx e^{-|j|/\ell(\beta)}$.
Note analogy to Aizenman-Molchanov fractional moment