Majority Vote Processes on Trees

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Majority Vote Process (MVP) - one of the classical interacting particle systems

Others include voter model, contact process, exclusion process

The MVP is the continuous time Markov process such that:

- $d \geq 1$
- $\epsilon \in [0, 1]$

1. On $\mathbb{Z}^d$, each site has "opinion" either 0 or 1.
2. Independently, at rate $1-\epsilon$ at each site, sites align w/ the majority opinion of their immediate neighbors.
3. Independently, at rate $\epsilon$ at each site, the opinion "flips".

Are primarily interested in case where $\epsilon$ is small, but positive.
How many equilibria does MVP have?
- Is open except in $d = 1$.
  Only 1 equilibrium (Gray (1992))

Compare with:

Voter Model with Noise (VMN) -
- Same as MVP, except sites randomly choose one of neighbors with which to align.

Not difficult to check:
- VMN converges expd quickly to its unique equilibrium.

Here, will investigate behavior of MVP on $d$-regular trees $T_d$.

\[ T_3 \]
Results will discuss:

\[ (B\text{-}Gray \ (2018)) \]

**Theorem 1.** For \( d \geq 5 \) and \( \epsilon > 0 \) small, there exist uncountably many mutually singular equilibria. \( \checkmark \) will spend more time on \( \checkmark \) less time on

**Theorem 2.** For any initial state close enough to such an equilibrium, process converges exp'ly quickly to this equilibrium.

**Comments:**

- Method does not depend on precise model. Includes, for example, stochastic Ising model on \( T_d \).

- Argument simplifies somewhat for oriented MVP, where above results are true for \( d \geq 4 \). \( \checkmark \) parents depend only on offspring

Ideas behind proof of Theorem 1

related ideas for Theorem 2, but more involved

Notation

\( (E_t^j)_{t \geq 0} \) is MVP w/ \( E_t^j(x) = 0 \) or \( 1 \) for \( x \in \mathbb{T}^d \)

Will consider special \( J \) defined using a tiling:

- Partition \( \mathbb{T}^d \) into even/odd sites.

- From each odd site, delete an edge to one of its offspring. For \( d \geq 7 \), can instead delete an edge from all sites.

- Each connected subset is a tile; corresponding collection is a tiling.

- \( J \) is compatible (wrt \( \alpha \) given tiling) if \( x, y \in T \) implies \( J(x) = J(y) \).

Note: For a given tiling,

1. there are uncountably many compatible \( J \).
Reasoning for Theorem 1

For \( d \geq 5 \) and small noise \( \varepsilon \), we will show that, for each tile \( T \),

\[
\lim_{n \to \infty} \frac{1}{\# T_n(x)} \sum_{y \in T_n(x)} | \mathbb{E}_{t_n^{-1}}(y) - T(y) | \leq 10\varepsilon \quad \text{a.s.}
\]

for each \( x \in T \) and \( t \).

Assume (2)

Let \( \mu_t^J = \text{Cesaro avg. over } [0, t] \text{ of measures of } \Xi_t^J \). One has

\[
\mu_{t_i}^J \xrightarrow{i \to \infty} \mu_t^J \text{ on some subsequence } t_i \to \infty,
\]

where \( \mu_t^J \) satisfies analog of (2).

Consequently, for \( J \neq J' \) (w/ both compatible wrt tiling),

(3) \( \mu_t^J \) and \( \mu_t^{J'} \) are mutually singular.

Theorem 1 follows from (1) and (3).
Demonstrate (2)

To demonstrate (2), construct a process $\delta_t$ on $\{0,1\}^\mathbb{N}$ s.t.

(4) $|\sum_{y}^{0} \delta_t(y) - \sum_{y}^{1} \delta_t(y)| \leq \delta_t(y)$ for all $y$

with

(2') $\lim_{n \to \infty} \frac{1}{#T_n(x)} \sum_{y \in T_n(x)} \delta_t(y) \leq 10\varepsilon$.

Construction of $\delta_t$:

- $\delta_0 \equiv 0$.
- $\delta_t(x) = 1$ at noise points $(x,t)$.
- $\delta_t$ behaves like $\delta_t^x$ at MV points $(x,t)$, except that neighborhood is now $T_t(x)$, and, for $\delta_t(x) = 1$ to occur, 1 (2) fewer 1's required if $x$ is even (odd).

These properties imply (4).

They also imply siblings evolve independently of one another.
To demonstrate (a'), set
\[ P_t = P(\delta_t(x) = 1) \quad \text{for } x \in T_d. \]

\( P_t \) depends on whether \( x \) is even or odd.
To simplify computations, instead delete an edge from all sites (not just odd).
This suffices for \( d \geq 7 \).

It follows that, for \( d \geq 7 \),
\[ P_t' \leq \epsilon (1 - P_t) + C_d P_t^2. \]

Even if \( \delta_t(z) = \delta_t(w) = 1 \),
still require \( \geq 2 \) sites
\( x \in T_d(x) \) \( w/ \delta_t(y) = 1 \)

Consequently,
\[ (5) \quad P_t' \leq \epsilon - P_t (1 - 2 C_d P_t). \]

Since \( P_0 = 0 \), for small \( \epsilon \)
\[ (6) \quad P_t \leq 2 \epsilon \quad \text{for all } t. \]
Since all sites in $T_n(x)$ evolve independently, (2') follows from (6) and SLLN.

**Basics behind proof of Theorem 2**

Proof of Theorem 2 is more complicated than that of Theorem 1.

Some ideas:

The limit

$$E_t^r \to E_\infty^r \quad \text{as } t \to \infty$$

is equivalent to

$$E_{s-t}^r \to E_\infty^r \quad \text{as } s \to \infty.$$  \hspace{1cm} \text{will show a.s. convergence in this setting}

process starts at time $-s$ instead of 0

Employ $s_{t-s}$ \hspace{1cm} analog of $s_t$, but starting at time $-s$

Since $s_{t-s}$ is increasing in $s$,

$$\lim_{s \to \infty} s_{t-s}$$ exists. \hspace{1cm} \text{want to also show for } s_0
Trace paths of possible influence backwards in time. Refer to as search histories.

Is complicated. No duality.

But can control these using $S_t^{-s}$.

Corresponding to these paths are supercritical branching processes, at least one of which must eventually die out, but not die out quickly if $E_0 S_0^x = E_0 S_0^x(x)$ for large $S_1 \leq S_2$.

Deaths are associated w/ noise pts; births w/ MV pts.

Unlikely event.

On path.

Paths of possible influence.

$s = 0$

$t = -s = -S_2$