Localization-delocalization transitions in random matrix models: a SPDE approach

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The localization-delocalization challenge

Understand the various facets of metal-insulator transitions in disordered systems, e.g. in the random matrix context:

**Power-Law Random Band Matrices (PRBM)**

Mirlin/Fyodorov/Dittes/Quezada/Seligman ’96

Real, symmetric $N \times N$ matrix $(H_{xy})$ with Gaussian rv’s as entries

\[
\mathbb{E} [H_{xy}] = 0
\]

\[
\mathbb{E} [H_{xy}^2] \sim \begin{cases} 
1 & \text{if } d(x, y) \leq 1 \\
 d(x, y)^{-2\alpha} & \text{else}
\end{cases}
\]

Simple limiting cases:

- **Random diagonal ensemble** $V = \text{diag}(V_x)$ with iid rv’s $(V_x)$
- **Gaussian orthogonal ensemble (GOE)** Real, symmetric $N \times N$ matrix $(\Phi_{x,y})$
  - $\Phi_{x,y}$ are centered Gaussian rv’s with
  \[
  \mathbb{E} [\Phi_{x,y}^2] = \frac{1 + \delta_{x,y}}{N}
  \]
  - $\Phi_{x,y}$ are independent for $x \leq y$.

Wigner, Dyson, Metha ≥’ 50
The localization-delocalization challenge

Predicted features – as a function of parameters in the model ($\alpha$) and/or energy:

- **Eigenvectors** $\psi$ undergo localization-delocalization transition

  Inverse participation ratios for $\ell^2$-normalized function: $\|\psi\|_\infty = \begin{cases} O(1) & \text{localization} \\ O(N^{-1/2}) & \text{delocalization} \end{cases}$

- **Eigenvalue statistics** changes from Poisson to Random Matrix (GOE)

  Rescaled random process of eigenvalues close to some energy $E$:

  $$\sum_{\lambda \in \sigma(H)} \delta_{N(\lambda-E)} \xrightarrow{N \to \infty} \begin{cases} \text{Poisson process} & \text{localization} \\ \text{GOE process} & \text{delocalization} \end{cases}$$
Rosenzweig-Porter Ensemble

\[ H(t) = V + \sqrt{t} \Phi \quad t \geq 0 \]

with \( N \times N \) GOE matrix \( \Phi \) and initial matrix \( V \) \hspace{1cm} (wlog \: V = \text{diag}(V_x))

Possible assumptions on random initial conditions:

A1 \hspace{0.5cm} V = \text{diag}(\omega_x) \text{ with iid } (\omega_x) \text{ with density } \varrho \in L^\infty, \hspace{0.5cm} \text{or}

A2 \hspace{0.5cm} V = A + \text{diag}(\omega_x) \text{ with } A = A^* \text{ and } (\omega_x) \text{ as in A1.}
Rosenzweig-Porter Ensemble

\[ H(t) = V + \sqrt{t} \Phi \quad t \geq 0 \]

with \( N \times N \) GOE matrix \( \Phi \) and initial matrix \( V \) \quad (wlog \ V = \text{diag}(V_x))

- Eigenvalues undergo **Dyson Brownian Motion (DBM)**

\[
d\lambda_{j}(t) = \sqrt{\frac{2}{N}} dB_j(t) + \frac{1}{N} \sum_{i \neq j} \frac{dt}{\lambda_j(t) - \lambda_i(t)}
\]

\[
\sqrt{t} \Phi_{x,y} \equiv \sqrt{\frac{1 + \delta_{xy}}{N}} B_{xy}(t)
\]

Dyson ’62

10 Trajectories until \( t = N^{-1} \) of 100-particle DBM with independent initial conditions

Trajectories until \( t = 1 \) of 10-particle DBM with independent initial conditions
Rosenzweig-Porter Ensemble

\[ H(t) = V + \sqrt{t} \Phi \quad t \geq 0 \]

with \( N \times N \) GOE matrix \( \Phi \) and initial matrix \( V \) \( \text{(wlog } V = \text{diag}(V_x)) \)

**Time scales** | **Results**
--- | ---
\( t \ll N^{-1} \) | pertubative regime | Soosten/W. ’17
\( N^{-1} \ll t \) | local equilibration regime | Erdős, Yau, …, Landon, Sosoe ≥ ’12
\( 1 \ll t \) | global equilibration regime delocalisation of eigenvectors | Erdős, Schlein, Yau, …, Bourgade …, Lee, Schnelli, … ≥ ’09
\( N^{-1} \ll t \ll 1 \) | intermediate regime | Soosten/W. ’17, Begnini ’17

Further reading: *Dynamical approach to random matrix theory* by Erdős and Yau.

Physics papers: Kravtsov/Khaymovich/Cuevas/Amini ’15, Facoetti/Vivo/Biroli ’16, Bogomolny/Sieber ’18
Spectral information

Green function: \[ G(x, z) = \langle \delta_x, (H - z)^{-1} \delta_x \rangle \quad x \in \{1, \ldots, N\}, \ z \in \mathbb{C}^+. \]

• Relation to normalized eigenfunctions \( \psi_j \) corresponding to ev \( \lambda_j \):
\[ G(x, z) = \sum_j \frac{|\psi_{\lambda_j}(x)|^2}{\lambda_j - z}. \]

• Upper bound on normalized eigenfunction with eigenvalue in \( I \):
\[ |\psi_{\lambda_j}(x)|^2 \leq \sup_{E \in I} \sum_k \frac{\eta^2}{(\lambda_k - E)^2 + \eta^2} |\psi_{\lambda_k}(x)|^2 = \eta \sup_{E \in I} \Im G(x, E + i\eta) \]

for any \( x \) and \( \eta > 0 \).

Stieltjes trafo of empirical eigenvalue measure: \[ S(z) = \frac{1}{N} \Tr (H - z)^{-1} \]

• Local density of states measure \( \mu \): \[ \int (x - z)^{-1} \mu(dx) = \mathbb{E} [S(z)] \]

• Rescaled eigenvalue process at \( E \) is captured by \( S(E + z/N) = \sum_j \frac{1}{N(\lambda_j - E) - z} \).
Flow of $S_t(z)$ under DBM

**Itô’s lemma** yields:  

$$dS_t(z) = \left[ S_t(z) \partial_z S_t(z) + \frac{1}{2N} \partial_z^2 S_t(z) \right] dt + dM_t(z)$$

with an (explicit) martingale term $dM_t(z)$.

**Viscous complex Burger’s equation**

**Lemma (Soosten/W. ’17)**

Assuming A2 for all $t \leq N^{-1+\varepsilon}$ with $\varepsilon > 0$ and $z \in \mathbb{C}^+$ ('Perturbative regime')

$$\mathbb{E} |S_t(z) - S_0(z)| \leq CN^{-\varepsilon/2} \left( 1 + \frac{1}{N \text{Im } z} + \frac{1}{(N \text{Im } z)^3} \right)$$

**Proof idea:** Use regularizing effect of the random potential on the drift, diffusion and martingale term through Wegner & Minami-type estimates.

Thus: **Rescaled eigenvalue process** remains Poisson of one started with Poisson as in A1.
Flow of $S_t(z)$ under DBM

Itô’s lemma yields:

$$dS_t(z) = \left[ S_t(z) \partial_z S_t(z) + \frac{1}{2N} \partial^2_z S_t(z) \right] dt + dM_t(z)$$

with an (explicit) martingal term $dM_t(z)$.

Viscous complex Burger’s equation

Inviscous case:

$$\partial_t S_t(z) = S_t(z) \partial_z S_t(z)$$

Method of characteristics:

Pastur flow

$$\dot{\xi}_t = -S_t(\xi_t), \quad \xi_0 = z \in \mathbb{C}^+$$

$$\frac{d}{dt} S_t(\xi_t(z)) = 0 \quad \text{i.e.} \quad \xi_t(z) = z - tS_0(z)$$

Example $V = 0$: $S_0(z) = -z^{-1}$ and hence $t S_t(w)^2 + w S_t(w) + 1 = 0$, i.e. semicircular law.
Local law: \( S_t(z) \) down to scale \( \text{Im} \, z \gg N^{-1} \)

\[
dS_t(z) = \left[ S_t(z) \partial_z S_t(z) + \frac{1}{2N} \partial_z^2 S_t(z) \right] dt + dM_t(z)
\]

For any \( z \in \mathbb{C} \) with \( \text{Im} \, z \geq \eta > 0 \), let \( \xi_t(z) \) be the random characteristics given by

\[
\dot{\xi}_s = -S_s(\xi_s), \quad \xi_0 = z \in \mathbb{C}^+
\]

stopped at \( \text{Im} \, \xi_t(z) = \eta/2 \).

**Theorem (Soosten/W. ’17)**

*For any \( z \in \mathbb{C} \) with \( \text{Im} \, z \geq \eta > 0 \):

\[
P \left( \sup_{|z| \leq \eta^{-1}} \sup_{s \in [0, t]} |S_s(\xi_s(z)) - S_0(z)| \geq \frac{6}{\sqrt{N\eta}} \right) \leq \frac{4N\eta}{\eta^{10}} e^{-N\eta/2}.
\]

**Key idea:** Integration trick & large deviation estimate for BM. (Blackboard)
Local law: \( S_t(z) \) down to scale \( \Im z \gg N^{-1} \)

Theorem (Soosten/W. '17)

For any \( z \in \mathbb{C} \) with \( \Im z \geq \eta > 0 \):

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\mathbb{P} \left( \sup_{|z| \leq \eta^{-1}} \sup_{s \in [0,t]} |S_s(\xi_s(z)) - S_0(z)| \geq \frac{6}{\sqrt{N\eta}} \right) \leq \frac{4N\eta}{\eta^{10}} e^{-N\eta/2} .
\]

Remaining task: Control & invert the characteristics \( \xi_s(z) \). This is done by assuming regularity of \( S_0 \) in some window \( W \subset \mathbb{R} \), i.e. for some \( K_u, K_l \in (0, \infty) \)

\[
K_l \leq \Im S_0(z) \leq |S_0(z)| \leq K_u
\]

uniformly for \( z \in W + [-K_u t, K_u t] + i[K_l t, 2] \).

Assumption is satisfied in case:

- if \( t \gg N^{-1} \) under assumption A1 with \( 0 < \inf_{v \in W} \rho(v) \) 
- if \( t \gg N^{-1/2} \) under assumption A2 with \( 0 < \inf_{z \in W} \mathbb{E} [\Im S_0(z)] \)
Excursion: Local laws for Schrödinger matrices

Consider $N \times N$ matrix of the form

$$ V = A + \text{diag}(\omega_1, \ldots, \omega_N) $$

with any $A = A^*$ non-random and iid $(\omega_x)$ with density $\varrho \in L^\infty$.

Lemma (Soosten/W. '18)

Let $z = E + i\eta$. Then for any $\mu > 0$

$$ \mathbb{P}(|S_0(z) - \mathbb{E}[S_0(z)]| > \mu) \leq C \exp(-c\mu^2 N\eta^2) $$

and

$$ \mathbb{P}(|\text{Im } S_0(z) > e\pi \|\rho\|_\infty + \mu) \leq \exp(-\mu N\eta). $$

Proof idea: McDiarmid concentration inequality + rank-one perturbation theory & spectral averaging.

(Blackboard)
Delocalization of eigenvectors

Green function satisfies an SDE

\[ dG_t(x, z) = \left( S_t(z) \partial_z G_t(x, z) + \frac{1}{2N} \partial_z^2 G_t(x, z) \right) dt + dM_t(x, z) \]

with an explicit martingal \( dM_t(x, z) \).

The method of characteristics suggests:

\[ G_t(x, \xi_t(z)) \approx G_0(x, z) = (V_x - z)^{-1} \]

and hence:

\[ \eta \text{ Im } G_t(x, E + i\eta) \approx \frac{\eta \text{ Im } \xi_t^{-1}(w)}{(V_x - \text{ Re } \xi_t^{-1}(E + i\eta))^2 + (\text{ Im } \xi_t^{-1}(E + i\eta))^2} \]

\[ \approx \begin{cases} C \frac{\eta}{t} & \text{if } |V_x - E| \leq Ct \\ 0 & \text{else.} \end{cases} \]
Intermediate regime

Theorem (Soosten/W. ’17)

Let \( t = N^{-1+\delta} \) with \( \delta \in (0, 1) \) and set \( \kappa > \delta > \theta \),

\[
X_\lambda = \{ x \in \{1, \ldots, N\} : |\lambda - V_x| > N^{-1+\kappa} \},
\]

and \( W \subseteq \text{supp} \ \varrho \). Then under assumption A1 there exists \( \gamma > 0 \) such that for any \( p > 0 \) and all sufficiently large \( N \) the \( \ell^2 \)-normalized eigenvectors in \( W \) carry only negligible mass inside \( X_\lambda \):

\[
\mathbb{P} \left( \sup_{\lambda \in \sigma(H_T) \cap W} \sum_{x \in X_\lambda} |\psi_\lambda(x)|^2 > N^{-\gamma} \right) \leq N^{-p}
\]

and are maximally extended outside \( X_\lambda \):

\[
\mathbb{P} \left( \sup_{\lambda \in \sigma(H_T) \cap W} \|\psi_\lambda\|_\infty > N^{-\theta/2} \right) \leq N^{-p}.
\]

Extension to deformed Wigner matrices: Begnini ’17
Outlook: Ultrametric ensemble

\( N \times N \) random matrices \((N = 2^n)\)

\[
H_n = \frac{1}{Z_{n,c}} \sum_{r=0}^{n} 2^{-(1+c) r} \sum_{B \in \mathcal{P}_r} \Phi_B
\]

with \( c \in \mathbb{R} \) and \((\Phi_B)\) independent GOE matrices.

- Normalization \( Z_{n,c} \) is chosen s.t. the variance matrix is doubly stochastic, i.e.
  \[
  \sum_y \mathbb{E} \left[ |\langle \delta_y, H_n \delta_x \rangle |^2 \right] = 1.
  \]

- Hierarchical analogue of PRBM \( \alpha = 1 \) corresponds to \( c = 0 \).

Results: partially confirming predictions by Fyodorov/Ossipov/Rodriguez '09

- \( c > 0 \): Localization regime and Poisson statistics
- \( c < -1/2 \): Delocalization regime \( \| \psi \|_\infty = \mathcal{O}(N^{-1/2}) \) and GOE statistics
- \( c \in (-1, -1/2) \): Infinite-volume operator has continuous spectrum.
Thank You!

P. von Soosten, S.W

- *Singular Spectrum and Recent Results on Hierarchical Operators*, arXive:1705.04884 (to appear in Contemp. Math.)