Compressible Navier-Stokes with non-monotone pressure and anisotropy

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Compressible Fluid dynamics

The compressible Navier-Stokes system reads

$$\partial_t \rho + \text{div}\,(u\,\rho) = 0,$$
$$\partial_t(\rho\,u) + \text{div}\,(\rho\,u \otimes u) - \mu\Delta u - (\lambda+\mu)\,\nabla\text{div}\,u = -\nabla p(\rho)$$

where $D\,u = (\nabla u + \nabla u^T)/2$.
written here in the barotropic case, in a bounded domain $\Omega \subset \mathbb{R}^d$ with for instance Dirichlet boundary conditions

$$u|_{\partial\Omega} = 0.$$

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$$\partial_t(\rho\, E(\rho, \theta) + \mathrm{div}\,(\rho\, u\, E) + \mathrm{div}\,(p\, u) = \mathrm{div}\,(\mathcal{S}\, u) + \mathrm{div}\,(\kappa(\theta)\,\nabla\theta).$$

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Variants of course exist. Some mostly fit within the same theory:

- With temperature for the Navier-Stokes-Fourier system

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or were inaccessible

• With density dependent viscosity (see nevertheless Bresch-Desjardins, Mellet-Vasseur...)

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• With temperature for the Navier-Stokes-Fourier system
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• With density dependent viscosity (see nevertheless Bresch-Desjardins, Mellet-Vasseur...)
• With non homogeneous viscosity given by a matrix A, or non local $p(\rho)$.
• With various type of pressure laws.

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For simplicity this talk deals only with the first system.

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Goal: Revisit the classical compactness theory by obtaining quantitative regularity estimates.

Other similar models

The same theory applies to many other models, for instance

$$\partial_t \rho + \operatorname{div}(u\,\rho) = 0,$$
$$-\Delta u = -\nabla p(\rho) + S.$$

In some applications to biology, $u = \nabla c$ with c the concentration of some chemical (or a sum of chemicals) used by the biological agents to interact.
Therefore, in those cases, $p(\rho)$ should include repulsive and attractive interactions.

What should the pressure law be?

It is a very old problem in physics...

- Ideal gas (Clapeyron 1834):

$$p = \rho\,\theta.$$

- Van der Waals law (1873):

$$(p + a\,\rho^2)\,(1 - b\,\rho) = c\,\rho\,\theta.$$

- Polynomial barotropic flows:

$$p = p(\rho), \quad \text{with often} \quad p = \rho^\gamma.$$

- Virial equation of state (H. Kamerlingh Onnes 1901):

$$p = \rho\,\theta\,\left(1 + B(\theta)\,\rho + C(\theta)\,\rho^2 + \ldots\right).$$

What should the pressure law be?

It is a very old problem in physics...

- Thermodynamically the stability of the equilibrium is directly connected to the monotonicity of p.
- Monotone laws are also required for hyperbolicity.
- However, many physical models have p non monotone.
- It is not clear why a thermodynamical assumption should control the stability of solutions over bounded times.
- The same type of questions may be asked about the stress tensors. For instance in some geophysical flows, one needs to take

$$\partial_t(\rho\, u) + \mathrm{div}\,(\rho\, u \otimes u) - \mu\,\Delta u - \mu_z\,\partial_{zz} u = -\nabla p(\rho),$$

with $\mu_z \neq 0$ (see for instance Temam-Ziane).

Which notion of solutions?

- Strong/classical solutions are of course the most convenient. They provide uniqueness and they preserve the most physical properties such as conservation of energy...
- However strong solutions only exist for short times, even in dimension 2 (vacuum problem), or for small initial pertubations of an equilibrium in some cases.
- Weak solutions can be global in time and also allow to work with non smooth initial data with only a bound on the energy.

State of the art: A priori estimates

Let us describe the available theory as developed first by P.L. Lions and extended by E. Feireisl. Start with the a priori estimates
Conservation of mass

$$\int \rho(t,x)\,dx = const.$$

Energy estimate For $P(\rho)$ s.t. $P'\,\rho - P = p(\rho)$,

$$\int \left(P(\rho(t,x)) + \frac{1}{2}\,\rho\,u^2\right)\,dx + \int_0^t \int |\nabla u|^2 = const.$$

Note that if $C^{-1}\rho^\gamma \leq p \leq C\,\rho^\gamma$ then $P(\rho) \sim \rho^\gamma$.

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Pressure estimates

$$\int_0^t \int \rho^a\,p(\rho)\,dx\,dt \leq C, \qquad a < \frac{2}{d}\,\gamma - 1.$$

Compactness of ρ

Take a sequence ρ_k, u_k of (approximate) solutions. u_k is compact in x and the only difficulty is the Compactness of ρ_k.

• P.L. Lions: Show that $w - \lim \rho_k \log \rho_k = A = \rho \log \rho$ with $w - lim \rho_k = \rho$.

$$\partial_t \rho_k \log \rho_k + \mathrm{div}\,(u_k\, \rho_k \log \rho_k) = -\mathrm{div}\, u_k\, \rho_k.$$

But

$$\mathrm{div}\, u_k = p(\rho_k) + \Delta^{-1}\mathrm{div}\,(\partial_t(\rho_k u_k) + \mathrm{div}\,(\rho_k\, u_k \otimes u_k)).$$

So

$$w - \lim \rho_k\, \mathrm{div}\, u_k = B + \rho\, \Delta^{-1}\mathrm{div}\,(\partial_t(\rho u) + \mathrm{div}\,(\rho\, u \otimes u)),$$

where $B = w - lim \rho_k\, p(\rho_k)$.

Compactness of ρ

Take a sequence ρ_k, u_k of (approximate) solutions.
Hence

$$\partial_t A + \text{div}\,(u\, A) = -B - \rho\, \Delta^{-1}(\partial_t(\rho u) + \text{div}\,(\rho\, u \otimes u)),$$

recalling $B = w - lim \rho_k\, p(\rho_k)$. While for $\tilde{B} = \rho\, w - lim p(\rho_k)$,

$$\partial_t \rho\, \log \rho + \text{div}\,(u\, \rho\, \log \rho) = -\tilde{B} - \rho\, \Delta^{-1}(\partial_t(\rho u) + \text{div}\,(\rho\, u \otimes u)).$$

Thus $A \leq \rho\, \log \rho$ and then $A = \rho\, \log \rho$ provided $B \geq \tilde{B}$.

Compactness of ρ

Take a sequence ρ_k, u_k of (approximate) solutions.
• Only gives compactness at the limit: No regularity estimates on ρ_k.
• The critical step is

$$w-\lim\ \rho_k\, p(\rho_k) \geq \rho\, w-\lim p(\rho_k),$$

which requires p increasing.
• Things are even more difficult with non-anisotropic stress tensors because

$$\operatorname{div} u_k = L\, p(\rho_k) + L\,\Delta^{-1} \operatorname{div}\left(\partial_t(\rho_k u_k) + \operatorname{div}\left(\rho_k\, u_k \otimes u_k\right)\right),$$

with L a non local operator of order 0, thus losing the pointwise relation between $\operatorname{div} u_k$ and $p(\rho_k)$.

Existence of weak solutions

The previous method yields

Theorem *P.L. Lions*
Assume $p'(\rho) \sim \rho^{\gamma-1}$ *with* $\gamma > 9/5$ *and p monotone.*
Then there exists a weak solution to compressible Navier-Stokes.

While with refined techniques

Theorem *E. Feireisl*
Assume $p'(\rho) \sim \rho^{\gamma-1}$ *with* $\gamma > 3/2$ *and p monotone.*
Then there exists a weak solution to compressible Navier-Stokes.

The idea

Propagate some explicit regularity on ρ by computing

$$\int \frac{|\rho(t,x)-\rho(t,y)|}{(|x-y|+h)^k}\,dx\,dy,$$

for some $k \geq d$.

However this corresponds to a Sobolev like regularity on ρ which cannot work. So instead...

The idea

Propagate some explicit regularity on ρ by computing

$$\int \frac{|\rho(t,x)-\rho(t,y)|}{(|x-y|+h)^k}\, W(t,x,y)\, dx\, dy,$$

for some $k \geq d$.
Where the weight W solves the same transport equation

$$\partial_t W + u(t,x)\cdot\nabla_x W + u(t,y)\cdot\nabla_y W = -D,$$

for a well chosen penalization D.

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Then explain that W cannot be too small, too often to bound

$$\int \frac{|\rho(x)-\rho(y)|}{(|x-y|+h)^k}\, dx\, dy,$$

in terms of h.

A new result

The main improvements are: No monotonicity assumption on p, explicit regularity.

Theorem

Assume that for $\gamma > 9/5$

$$\frac{\rho^\gamma}{C} \leq p(\rho) \leq C\,\rho^\gamma, \quad |p'(\rho)| \leq C\,\rho^{\gamma-1}.$$

Then there exists a weak solution to compressible Navier-Stokes. Moreover the solution satisfies for any $k > d$

$$\int_{\rho(x),\ \rho(y)>\eta} \frac{|\rho(x)-\rho(y)|}{(|x-y|+h)^k}\,dx\,dy \leq C_\eta\,\frac{h^{-(k-d)}}{|\log h|^\mu},$$

for some $\mu > 0$.

The new result in the anisotropic case

Theorem

Assume $p(\rho) \sim \rho^\gamma$ *with* $\gamma > \bar\gamma$ *and that* A *is smooth with*

$$\|A(x) - \mu\, I\|_{L^\infty} < \mu\, C_*,$$

for some universal constant C_*. *Then there exists a weak solution to the compressible Navier-Stokes*

$$\partial_t \rho + div\,(u\,\rho) = 0,$$
$$\partial_t(\rho\, u) + div\,(\rho\, u \otimes u) - div\,(A(x)\, D\, u) - \lambda\, \nabla div\, u = -\nabla p(\rho),$$

where $D\,u = (\nabla u + \nabla u^T)/2$.

Sketch of the proof-Weighted norms Part 1

Denote $\delta\rho = \rho(t,x) - \rho(t,y)$ and observe that

$$\partial_t|\delta\rho|^2 + \operatorname{div}_x(u(t,x)\,|\delta\rho|^2) + \operatorname{div}_y(u(t,y)\,|\delta\rho|^2)$$
$$= -\frac{1}{2}\,(\operatorname{div} u(x) - \operatorname{div} u(y))\,\delta\rho\,(\rho(x)+\rho(y)).$$

Recall

$$\partial_t W + u(t,x)\cdot\nabla_x W + u(t,y)\cdot\nabla_y W = -D,$$

and calculate

$$\frac{d}{dt}\int \frac{|\delta\rho|^2}{(|x-y|+h)^k}\,W = -\frac{1}{2}\int \frac{\operatorname{div} u(x) - \operatorname{div} u(y)}{(|x-y|+h)^k}\,\delta\rho\,(\rho(x)+\rho(y))\,W$$
$$+\int \frac{(u(x)-u(y))\cdot(x-y)}{|x-y|\,(|x-y|+h)^{k+1}}\,|\delta\rho|^2\,W - \int \frac{|\delta\rho|^2}{(|x-y|+h)^k}\,D.$$

Sketch of the proof-Weighted norms Part 2

Use the momentum equation to bound

$$
\begin{aligned}
|\mathrm{div}\, u(x) - \mathrm{div}\, u(y)| &= |p(\rho(x)) - p(\rho(y)) + OK| \\
&\leq C\, (\rho^{\gamma-1}(x) + \rho^{\gamma-1}(y))\, \delta\rho + OK.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&-\frac{1}{2}\int \frac{\mathrm{div}\, u(x) - \mathrm{div}\, u(y)}{(|x-y|+h)^k}\, \delta\rho\, (\rho(x)+\rho(y)) \\
&\qquad \leq C \int \frac{|\delta\rho|^2}{(|x-y|+h)^k}\, (\rho^{\gamma}(x)+\rho^{\gamma}(y))\, W + OK.
\end{aligned}
$$

Observe that if $p(\rho)$ is increasing then $\delta\rho(p(\rho(x)) - p(\rho(y))) \geq 0$ and the corresponding terms do not need to be controlled.

Sketch of the proof-Weighted norms Part 3

Use the classical inequality

$$|u(x) - u(y)| \leq C\,(M|\nabla u|(x) + M|\nabla u|(y))\,|x - y|,$$

with M the maximal operator. This implies

$$\int \frac{(u(x) - u(y)) \cdot (x - y)}{|x - y|\,(|x - y| + h)^{k+1}}\,|\delta\rho|^2\,W \leq C \int \frac{|\delta\rho|^2}{(|x - y| + h)^k}\,(M|\nabla u|(x) + M|\nabla u|(y))\,W.$$

Sketch of the proof-Weighted norms conclusion

Summing up, one finds

$$\frac{d}{dt}\int \frac{|\delta\rho|^2}{(|x-y|+h)^k}\,W$$
$$\leq \int \frac{|\delta\rho|^2}{(|x-y|+h)^k}\,\left(C\,(M|\nabla u|(x)+\rho^\gamma(x)+sym.)\,W-D\right)\leq 0,$$

if one takes

$$D=C\,(M|\nabla u|(x)+M|\nabla u|(y)+\rho^\gamma(x)+\rho^\gamma(y))\,W.$$

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if one takes

$$D=C\,(M|\nabla u|(x)+M|\nabla u|(y)+\rho^\gamma(x)+\rho^\gamma(y))\,W.$$

Instead we take $W(x,y)=w(x)+w(y)$ with

$$D=C\,(M|\nabla u|(x)+\rho^\gamma(x))\,w(x)+sym,$$

and things are much more complicated...

Sketch of the proof- The compactness

Assume that one has

$$\int \frac{|\delta\rho|}{(|x-y|+h)^k}\,(w(x)+w(y)) \leq C + \ldots,$$

with

$$\partial_t w + u(x)\cdot\nabla w = -C\,(M|\nabla u|(x) + \rho^\gamma(x))\,w(x), \quad w(t=0)=1.$$

Now calculate

$$\frac{d}{dt}\int \rho(x)\,|\log w(x)|\,dx = C\int \rho(x)\,(M|\nabla u|(x)+\rho^\gamma(x))\,dx \leq C.$$

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with

$$\int \rho(x)\,|\log w(x)|\,dx \leq C.$$

That implies

$$\int \frac{|\delta\rho|}{(|x-y|+h)^k} \leq \frac{C}{h^{k-d}\,|\log h|} + \ldots$$

Some of the additional difficulties

- All the estimates must be delocalized as one cannot control $M\,|\nabla u_k|(x)\,w(y)$ by $M\,|\nabla u_k|(x)\,w(x)$. Instead one uses

$$|u_k(x)-u_k(y)|\leq C\int_{|z-x|\leq 2\,|x-y|}\frac{|\nabla u_k(z)|}{|z-x|^{d-1}}\,dz+sym.$$

- The penalization are more complicated as the current ones would require $\rho\in L^{\gamma+1}$, for instance

$$D=\lambda(\rho^{p-1}\,|\mathrm{div}\,u|+M\,|\nabla u|+\rho^{\gamma})\,w(x)+sym.$$

Some of the additional difficulties-2

- Delocalization is achieved through square function or their equivalent in Besov spaces. Thus one actually controls

$$\int \frac{|\delta\rho|}{(|x-y|+h_0)^d}\,(w(x)+w(y)) \sim \int_{h_0}^1 h^{k-d} \int \frac{|\delta\rho|}{(|x-y|+h_0)^k}\,(w(x)+w(y))\,\frac{dh}{h},$$

with the property that for a normalized convolution kernel K_h

$$\int_{h_0}^1 \frac{dh}{h}\,\|K_h\star u - K_h\star u(.+h\,\omega)\|_{L^p} \leq C\,|\log h_0|^{1/2}\,\|u\|_{L^p}.$$

Extension: Viscosity and numerical methods

- For numerical or construction purposes, it is interesting to consider

$$\partial_t \rho + \operatorname{div}(\rho\, u) = \varepsilon\, \Delta \rho.$$

- Some cases with temperature are identical, others are still open for example the full virial.
- The result in the anisotropic case should certainly be improved...