Nonlocal interaction PDEs with nonlinear diffusion

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A discrete particle system

- N particles, located at $X_1(t), \ldots, X_N(t) \in \mathbb{R}^d$ with masses $m_1, \ldots, m_N$.
- Subject to *binary interaction forces* depending on their position.
- *Friction dominated* regime: no inertia.

$$\frac{dX_j(t)}{dt} = -\sum_{k \neq j} m_k \nabla G(X_j(t) - X_k(t)), \qquad j = 1, \ldots, N. \tag{1}$$

Typical assumptions for the interaction potential G

- $G \in C(\mathbb{R}^d)$, with $G(0) = 0$,
- Radial symmetry $G(x) = g(|x|)$,

Notation: g increasing $\Rightarrow$ G *attractive*, g decreasing $\Rightarrow$ G *repulsive*.

Stochastic version:

$$dX_j(t) = -\sum_{k \neq j} m_k \nabla G(X_j(t) - X_k(t))dt + \sigma_N dW^j(t)$$

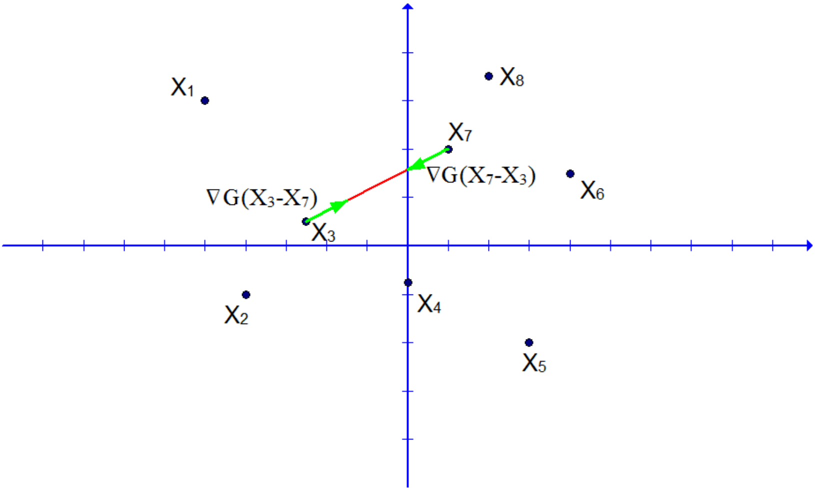

Figure: N interacting particles

Main motivation: population dynamics

Animal swarming:

- Okubo (1980)
- Oelschläger (1989)
- Morale, Capasso, and Oelschläger (1998)
- Mogilner, Edelstein-Keshet (1999)
- Topaz, Bertozzi, and Lewis (2006)

Typical interaction potentials:

- attractive-repulsive *Morse* potentials $G(x) = -C_a e^{-|x|/l_a} + C_r e^{-|x|/l_r}$
- combination of *Gaussian* potentials $G(x) = -C_a e^{-|x|^2/l_a} + C_r e^{-|x|^2/l_r}$
- smoothed characteristic functions of a set $G(x) = \alpha\delta_\epsilon * \chi_A(x)$.

Hydrodynamic $N \to +\infty$ limit

Empirical measure:

$$\mu_N(t) = \left(\sum_{j=1}^{N} m_j \right)^{-1} \sum_{k=1}^{N} m_k \delta_{X_k(t)}$$

Formal limit of μ_N in the stochastic case

Assuming $\lim_{N \to +\infty} \sigma_N = \sigma > 0$, then

$$\frac{\partial \mu}{\partial t} = \frac{\sigma^2}{2} \Delta \mu + \mathrm{div}(\mu \nabla G * \mu)$$

Distributional PDE for μ_N for $\sigma = 0$

$$\frac{\partial \mu}{\partial t} = \mathrm{div}(\mu \nabla G * \mu)$$

More motivations: Interplay with physics

Mean-field limits of large particle systems in statistical mechanics:

- Onsager (1949) - Vortex dynamics
- Morrey (1955) - Derivation of hydrodynamics from statistical mechanics
- Dobrushin (1993) - Vlasov equation
- Golse (2003) - Review paper

In those contexts, the potential G *blows-up at the origin*, which renders the rigorous analytical framework of the model a challenging issue.
Kinetic modeling for *granular media*:

- Benedetto, Caglioti, Pulvirenti (1997)
- Brilliantov, Pöschel (2004)
- Toscani (2004)

Here, G is a convex attractive potential, typically $G(x) = |x|^\alpha$ with $\alpha > 1$.

More motivations: chemotaxis

- In many problems in biology, such as the 2*d* Keller-Segel model

$$\partial_t \rho = \Delta\rho + \frac{\chi}{2\pi}\mathrm{div}(\rho\nabla \log|\cdot| * \rho),$$

the dichotomy between the *repulsive* linear diffusion term and the *attractive log* 'chemotaxis' term produces *blow-up* (concentration) of solutions in finite time. No one knows (up to now) how to define solutions in a *measure* sense after blow up.

- The large time behavior for models with 'milder' aggregation force and with nonlinear diffusion

$$\partial_t \rho = \Delta\rho^m + \mathrm{div}(\rho\nabla G * \rho)$$
$$G(x) = g(|x|), \qquad g'(r) > 0, \qquad G \in W^{1,\infty},$$

is a (most of the times) highly nontrivial question.

Simplification: no diffusion. Measure solutions theory (particles remain particles).

More motivations:

- Alignment of actin laments with or without cross-linking proteins, cf. Kang, Perthame, Primi, Stevens, Velazquez (2009). G *double well* potential.
- Kinetic dithering
$$\partial_t \rho = -\mathrm{div}(\rho \nabla (G * (\rho - \sigma)))$$
with $\sigma \in L^1_+$ being a given profile, and $\int \rho = \int \sigma$. Typically, $G(x) = |x|^\alpha$. Stationary solution $\rho = \sigma$. Stable for large times? PhD thesis of J.-C. Hütter. Ref: Fornasier, Haškovec, Steidl - 2012.
- Opinion formation: Sznajd-Weron, Sznajd (2000) - Aletti, Naldi, Toscani (2007). Quasi invariant opinion limit of kinetic models.
- Crowd movements: Helbing's social force modelled via nonlocal forces, cf. Hughes (2002), Cristiani et al. (2011), Colombo et al. (2012).

Mathematical motivation

- Models with nonlocal attractive-repulsive kernels

  $$\partial_t \rho = \mathrm{div}(\rho \nabla G * \rho)$$

  with G being a *double-well* potential, e. g. Lennard–Jones. Stationary solutions? How do they look like?

  - Fetecau, Huang, Kolkolnikov - 2011: L^1 stationary states.
  - von Brecht, Bertozzi - 2012: aggregation sheets.
  - Balagué, Carrillo, Laurent, Raoul - 2011: radial ins/stability of 'spherical shells'.

- Similarities with $2d$ incompressible Euler.

- A repulsive nonlocal approximation for nonlinear diffusion

  $$\frac{\partial \rho}{\partial t} = \mathrm{div}(\rho \nabla G_\epsilon * \rho)$$

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What is a gradient flow?

Given a smooth function $F : \mathbb{R}^d \to \mathbb{R}$, a differentiable curve $[0,+\infty) \ni t \mapsto X(t) = \mathbb{R}^d$ is a gradient flow of F if $X(t)$ satisfies

$$\dot{X}(t) = -\nabla F(X(t)).$$

- Energy dissipation:

$$\frac{d}{dt} F(X(t)) = -|\nabla F(X(t))|^2$$

- Implicit Euler variational derivation: time step $\tau > 0$, $X_\tau(t) = X_\tau^n$ for $t \in ((n-1)\tau, n\tau]$, with

$$X_\tau^n = \mathrm{argmin}\{\frac{1}{2\tau}|X - X_\tau^n|^2 + F(X),\ X \in \mathbb{R}^d\}$$

- $D^2 F \geq \lambda \mathbb{I}$ implies stability

$$\frac{d}{dt}|X_1(t) - X_2(t)|^2 = -2 < X_1(t) - X_2(t), \nabla F(X_1(t)) - \nabla F(X_1(t)) >$$
$$\leq -2\lambda |X_1(t) - X_2(t)|^2.$$

Gradient flow structure of the ODE particle system

Consider

$$\frac{dX_j(t)}{dt} = -\sum_{k\neq j} m_k \nabla G(X_j(t) - X_k(t)), \qquad j = 1, \ldots, N.$$

with $G(-x) = G(x)$ and $G \in C^2(\mathbb{R}^d)$.

Weighted metric structure

Denote $\mathbf{m} = (m_1, \ldots, m_N)$. For $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{dN}$, let

$$< \mathbf{X}, \mathbf{Y} >_{L^2_{\mathbf{m}}} := \sum_{j=1}^{N} m_j X_j Y_j, \qquad \|\mathbf{X}\|^2_{L^2_{\mathbf{m}}} = < \mathbf{X}, \mathbf{X} >_{L^2_{\mathbf{m}}} .$$

Frechét differential

Let $\mathbf{F} \in C^1(\mathbb{R}^{dN})$. The linear operator $\mathrm{grad}_{\mathbf{X}} \mathbf{F}[X]$ is defined by

$$\lim_{\epsilon\to 0} \frac{\mathbf{F}[\mathbf{X} + \epsilon \mathbf{Y}] - \mathbf{F}[\mathbf{X}]}{\epsilon} =:< \mathrm{grad}_{\mathbf{X}} \mathbf{F}[\mathbf{X}], \mathbf{Y} >_{L^2_{\mathbf{m}}} = \sum_{j=1}^{N} m_j \nabla_{X_j} \mathbf{F}[\mathbf{X}] \cdot Y_j.$$

Gradient flow structure of the ODE particle system

Energy functional

Let $\mathbf{X} := (X_1, \ldots, X_N)^T$.

$$\mathbf{G}[\mathbf{X}] := \frac{1}{2} \sum_{i,j} m_i m_j G(X_i - X_j)$$

Then

$$\dot{\mathbf{X}}(t) = -\mathrm{grad}_{\mathbf{X}} \mathbf{G}[\mathbf{X}(t)]. \tag{2}$$

Problem (2) makes sense if $G \in C^1(\mathbb{R}^d)$.

Regularity and collisions

If $G \in C^2(\mathbb{R}^d)$, then particles do not collide.

Mildly singular, locally attractive kernels

Assume

(K1) $G(-x) = G(x)$

(K2) $G \in C^1(\mathbb{R}^d \setminus \{0\})$

(K3) G has a local minimum at $x = 0$

(K4) G is λ-convex, i. e. $G(x) - \frac{\lambda}{2}|x|^2$ is convex on $\mathbb{R}^d$.

Examples:

- Morse type potentials $G(x) = -e^{-a|x|}$, with $a > 0$,
- *Pointy* potentials, i. e. with a Lipschitz point at the origin,
- Power laws $G(x) = |x|^\alpha$ with $\alpha \in (1, 2)$, cf. [Li, Toscani - 2004], [Burger, DF - 2008]

Kernels with above assumptions (K1)–(K4) possibly produce *finite time collapse* $\mu = \delta_{x_c}$, with x_c center of mass of the particles (constant in time).

Weaker gradient flow structure

Introduce the sub-differential of G

$$\partial G(x) := \left\{k \in \mathbb{R}^d \,:\, G(y) - G(x) \geq k \cdot (y - x) + o(|x - y|) \text{ for all } y \in \mathbb{R}^d\right\},$$

and the minimal sub-differential of G

$$\partial^0 G(x) = \mathrm{argmin}_{k \in \partial G(x)} |k| = \begin{cases} \nabla G(x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Sub-differential structure of $L^2_{\mathbf{m}}$

$$\partial \mathbf{G}[\mathbf{X}] =:= \left\{K \in L^2_{\mathbf{m}} \,:\, \mathbf{G}(\mathbf{Y}) - \mathbf{G}(\mathbf{X}) \geq < K, (\mathbf{Y} - \mathbf{X}) >_{L^2_{\mathbf{m}}} + o(\|\mathbf{X} - \mathbf{Y}\|_{L^2_{\mathbf{m}}}) \text{ for all } Y \in L^2_{\mathbf{m}}\right\}.$$

Weaker gradient flow structure

We replace our particle system with

$$\frac{dX_j(t)}{dt} \in -\sum_{k\in C_j(t)} m_k \partial^0 G(X_j(t)-X_k(t)), \quad C_j(t)=\{k:\ X_j(t)\neq X_k(t)\}. \quad (3)$$

Then, it is easily checked that

$$\dot{\mathbf{X}}(t) \in -\partial^0 \mathbf{G}[\mathbf{X}(t)], \quad (4)$$

with $\partial^0 \mathbf{G} = \mathrm{argmin}_{K\in\partial\mathbf{G}} \|K\|_{L^2_{\mathbf{m}}}$.

Well posedness in the discrete case

- λ-convexity of the functional $\mathbf{G}$
- Existence and uniqueness of gradient flows.

Finite time collapse for attractive potentials

Assume G satisfies (K1)–(K4) and the additional conditions

$$G(x) = g(|x|), \quad g'(r) > 0^1 \text{ for } r > 0, \quad \frac{g'(r)}{r} \text{ non-increasing.} \tag{5}$$

Proposition (Finite time collapse)

Let $X_1, \ldots, X_N$ evolve according to (4), i. e.

$$\dot{X}_j(t) = -\sum_{X_k(t) \neq X_j(t)} m_k \nabla G(X_j(t) - X_k(t)).$$

Then, all the particles collapse in a finite time, i. e. $X_j(t) = \delta_{C_m}$ for all $t \geq t^*$ for some t^*, iff

$$\int_0^\varepsilon \frac{1}{g'(z)} dz < +\infty \tag{6}$$

for some $\varepsilon > 0$.

[1] G is called *attractive* when $g'(r) > 0$ and *repulsive* when $g'(r) < 0$

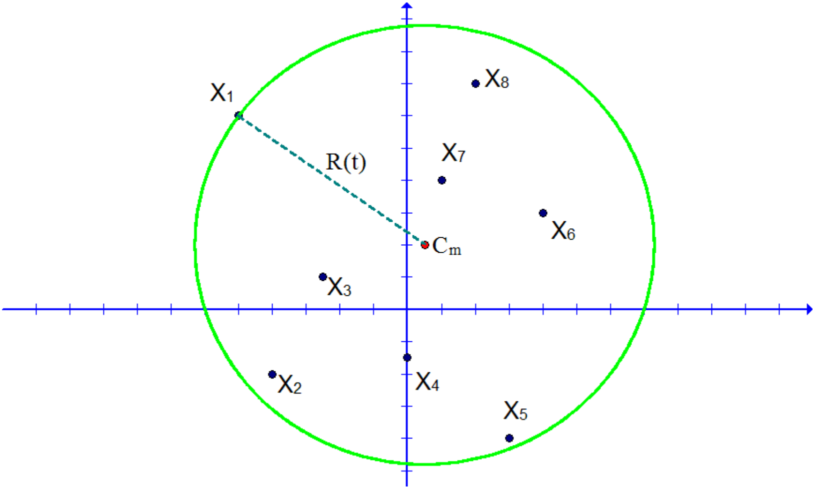

Figure: The quantity $R(t) = \max\{|X_j(t) - C_m|,\ j = 1, \ldots, N\}$.

Proof

Assume $\sum_{j=1}^N m_j = 1$. Center of mass $C_m = \sum_{j=1}^N m_j X_j(t)$ is preserved. Assume for simplicity $C_m = 0$.

$$\frac{d}{dt}R(t) = \frac{d}{dt}|X_1(t)| = -\frac{X_1(t)}{|X_1(t)|}\cdot\sum_{j\neq 1} m_j \nabla G(X_1(t) - X_j(t))$$
$$= -\sum_{j\neq 1} m_j X_1(t)\cdot(X_1(t) - X_j(t))\frac{g'(|X_1(t) - X_j(t)|)}{|X_1(t)||X_1(t) - X_j(t)|}.$$

Since $X_1(t)\cdot X_j(t) \leq |X_1(t)|^2$, and since $g'(r)/r$ is non increasing, we use $|X_1(t) - X_j(t)| \leq 2|X_1(t)|$:

$$\frac{d}{dt}R(t) \leq -\frac{g'(2|X_1(t)|)}{2|X_1(t)|^2}\sum_{j\neq 1} m_j\left(|X_1(t)|^2 - X_1(t)\cdot X_j(t)\right)$$
$$= -(1-m_1)g'(2|X_1(t)|) + \frac{g'(2|X_1(t)|)}{2|X_1(t)|^2}X_1(t)\cdot(-m_1 X_1(t)) = -g'(2R(t))$$

and the assertion is proven. Notice that the collapse time is independent of N.

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Ingredients for the continuum theory[2]

Aim: produce a unique notion of *measure* solution for

$$\frac{\partial \mu}{\partial t} = \mathrm{div}(\mu \nabla G * \mu).$$

The measure space

$$\mu \in \mathcal{P}_2(\mathbb{R}^d) := \left\{ \mu \in \mathcal{P}(\mathbb{R}^d), \ \int |x|^2 d\mu(x) < +\infty \right\}$$

endowed with the 2-*Wasserstein* distance

$$d_2(\mu,\nu)^2 = \inf \left\{ \iint_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 d\gamma(x,y), : \gamma \in \Gamma(\mu,\nu) \right\}$$

$$\Gamma(\mu,\nu) = \left\{ \gamma \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) \ : \ \mu \text{ and } \mu \text{ are the marginals of } \gamma \right\}$$

The functional

$$\mathcal{G}[\mu] = \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} G(x-y) d\mu(x) d\mu(y)$$

[2] Ambrosio, Gigli, Savaré - Birkhäuser 2005

Why the Wasserstein distance?

Go back to the discrete case:

$$\mu := \sum_{j=1}^{N} m_j X_j, \quad \nu := \sum_{j=1}^{N} m_j Y_j.$$

The *natural* distance is

$$d(\mu,\nu)^2 = \inf\left\{ \int_0^1 \| \frac{d}{ds}\mathbf{X}(\cdot)\|^2_{L^2_{\mathbf{m}}},\ X_j(0) = X_j,\ X_j(1) = Y_j \right\}.$$

The natural *continuum* version is:

$$d(\mu,\nu)^2 = \inf\left\{ \int_0^1 \int |v_s(x)|^2 d\mu_s(x),\ \partial_s\mu_s + \mathrm{div}(\mu_s v_s)0,\ \mu_0 = \mu,\ \mu_1 = \nu \right\},$$

which coincides with the 2–Wasserstein distance according to the Benamou-Brenier formula.

Definition of Wasserstein gradient flow

An absolutely continuous curve $[0,+\infty) \ni t \mapsto \mu(t) \in \mathcal{P}(\mathbb{R}^d)$ is a *Wasserstein gradient flow* of the functional $\mathcal{G}$ iff

$$\frac{\partial \mu(t)}{\partial t} + \mathrm{div}(\mu(t)v(t)) = 0, \qquad \text{in } \mathcal{D}'(\mathbb{R}^d \times [0,+\infty))$$
$$v(t) = -\partial^0 G * \mu(t) = -\int_{x\neq y} \nabla G(x-y)d\mu(y,t).$$

Notice that $\partial^0 G * \mu(t)$ coincides with the minimal sub-differential of $\mathcal{G}$ on $\mathcal{P}_2(\mathbb{R}^d)$, namely

$$\partial^0 G * \mu(t) = \mathrm{argmin}_{\mathbf{v}\in\partial\mathcal{G}[\mu]} \|\mathbf{v}\|_{L^2(d\mu:\mathbb{R}^d)}$$
$$\partial\mathcal{G}[\mu] = \Big\{ \mathbf{v} \in L^2(d\mu) \ :$$
$$\mathcal{G}[\nu] - \mathcal{G}[\mu] \geq \inf_{\gamma_o \in \Gamma(\mu,\nu)} \iint_{\mathbb{R}^d\times\mathbb{R}^d} \mathbf{v}(x)\cdot(y-x)d\gamma_o(x,y) + o(d_2(\mu,\nu)) \Big\},$$
$$\gamma_o \in \Gamma(\mu,\nu) \quad \text{such that} \quad d_2(\mu,\nu) = \iint_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 d\gamma_o(x,y).$$

Existence and uniqueness of solutions

Theorem (Existence and uniqueness[a])

[a] Carrillo, DF, Figalli, Laurent, Slepcev - Duke Math. J. - 2011

- *Let* $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$. *Then, there exists a unique Wasserstein gradient flow solution for* $\mathcal{G}$ *with* μ_0 *as initial datum. Moreover,*

$$\mathcal{G}[\mu(t)] + \int_0^t ds \int_{\mathbb{R}^2} \left|\partial^0 G * \mu(x,s)\right|^2 d\mu(x,s) \leq \mathcal{G}[\mu_0], \tag{7}$$

  for all $t \geq 0$.

- *Let* $\mu_1^0, \mu_2^0 \in \mathcal{P}_2(\mathbb{R}^d)$. *Let* $\mu_1(t)$ *and* $\mu_2(t)$ *be Wasserstein gradient flows for* $\mathcal{G}$ *with* μ_1^0 *and* μ_2^0 *as initial data respectively. Then,*

$$d_2(\mu_1(t), \mu_2(t)) \leq e^{|\lambda| t} d_2(\mu_1^0, \mu_2^0), \tag{8}$$

  for all $t \geq 0$.

Finite time collapse for general solutions

Theorem (Finite total collapse[a])

[a] Carrillo, DF, Figalli, Laurent, Slepcev - Duke Math. J. - 2011

Let $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ *compactly supported. Let* $\mu(t)$ *the corresponding gradient flow of* $\mathcal{G}$*. Let*

$$C_m := \int_{\mathbb{R}^d} x d\mu(x,t).$$

Then, there exists a time t^* *depending only on the radius of* $\mathrm{spt}(\mu_0)$ *such that*

$$\mu(t) = \delta_{C_m},$$

for all $t \geq t^*$*.*

Proof

Similar to an old idea of R. Dobrushin (1979).

1. *Atomization* of μ_0: for a fixed arbitrary $\varepsilon > 0$, take $\mu_0^N = \sum_{j=1}^N m_j \delta_{X_j}$ such that
$$d_2(\mu_0, \mu_0^N) \leq \varepsilon.$$
2. Let the particles $X_1, \ldots, X_N$ evolve via the discrete particle system. Let t^* be the collapse time,
$$X_1(t) = \ldots = X_N(t) = C_m, \quad \text{for all } t \geq t^*.$$
3. This means that $\mu^N(t) := \sum_{j=1} N m_j \delta_{X_i(t)} = \delta_{C_m}$ for all $t \geq t^*$.
4. The stability property (8) implies
$$d_2(\mu(t^*), \mu^N(t^*)) \leq e^{-\lambda t^*} d_2(\mu_0, \mu_0^N) \leq \varepsilon e^{-\lambda t^*},$$
which is an arbitrary small quantity. Hence,
5. $\mu(t^*) = \mu^N(t^*) = \delta_{C_m}$.

Global confinement for attractive-repulsive potentials[3]

Assume G as in (K1)–(K4), plus

(K5) $G(x) = g(|x|)$, $g \in C^1((0,+\infty))$,

(K6) $g'(r) > 0$ for $r > R_a$ for some $R_a > 0$,

(K7) $g'(r) > -C_G$ for $r < R_a$ for some $C_G > 0$.

Moreover, assume either

(K8) there exists $\bar{R} > 0$ such that $g'(r) \geq 4C_G$ for all $r \geq \bar{R}$,

or

(K9) $\liminf_{r\to 0} g(r) > -\infty$, and $\lim_{r\to+\infty} g'(r)\sqrt{r} = +\infty$.

Then, there exists $R^* > 0$ depending only on G and on μ_0 such that

$$\mathrm{spt}(\mu(t)) \subset B(0,R^*), \qquad \text{for all } t \geq 0.$$

Remark: conditions (K5)–(K7) alone are not enough for global confinement (Theil, 2006).

[3] Carrillo, DF, Figalli, Laurent, Slepcev - Nonlinear Anal. - 2012

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N–dependent repulsion range[4]

$$\frac{dX_j(t)}{dt} = -\sum_{k\neq j} m_k \nabla G(X_j(t) - X_k(t)) - \sum_{k\neq j} m_k \nabla V_N(X_j(t) - X_k(t)), \; j = 1, \ldots, N$$

$$V_N(x) = N^{d\beta} V(N^\beta x), \quad \beta \in (0,1)$$

$$V(x) = v(|x|), \quad v \in C^2((0,+\infty)), \quad v'(r) < 0, \text{ as } r > 0,$$

$$V \geq 0, \quad \int_{\mathbb{R}^d} V(x)dx = \varepsilon.$$

- V_N is a *repulsive* kernel, with a *range of interaction* $O(N^{-\beta})$ and *strength of the interaction force* $O(N^{d\beta})$ depending on the number of individuals N.
- Formally $V_N(x) \to \varepsilon\delta$ in $\mathcal{D}'$ as $N \to +\infty$.

Formal limit of the particle system

$$\frac{\partial \mu}{\partial t} = \mathrm{div}(\mu \nabla G * \mu) + \varepsilon \mathrm{div}(\mu \nabla \mu). \tag{9}$$

Hence... a quadratic *porous medium* type diffusion term appears.

[4] Ölschläger - Prob. Th. Rel. Fields - 1989

Basic properties of the limiting equation

Assume

- $G(x) = g(|x|)$, $g \in C^2([0, +\infty))$,
- $g'(r) > 0$ for all $r > 0$,
- $\mathrm{spt}\, G = \mathbb{R}^d$, $G \leq 0$, $G \in L^1(\mathbb{R}_d)$.

Regularizing effect

For all initial data $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$, the corresponding solutions are *densities*, $\mu(t) = \rho(t) d\mathcal{L}_d$.

Conservation of the center of mass

Let

$$CM[\rho(t)] := \int x\rho(x,t)dx,$$

then $CM[\rho(t)] = CM[\rho_0]$ for all $t \geq 0$.

Wasserstein gradient flow for the limiting equation

$$\frac{\partial \rho}{\partial t} = \mathrm{div}(\rho \nabla(\varepsilon \rho + G * \rho)).$$

Energy functional:

$$E[\rho] := \frac{\varepsilon}{2} \int_{\mathbb{R}^d} \rho^2(x) dx + \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(x-y)\rho(y)\rho(x) dy dx. \tag{10}$$

Energy identity:

$$E[\rho(t)] + \int_0^T \int_{\mathbb{R}^d} \rho \left| \nabla(\varepsilon \rho + G * \rho) \right|^2 dx dt = E[\rho_0]. \tag{11}$$

The identity (11) can be proven rigorously in the context of the *Wasserstein gradient flow* theory developed in [Ambrosio, Gigli, Savaré, Birkhäuser 2003].

A key question: large time behavior

How does $\rho(t)$ behave as $t \to +\infty$? There are (basically) three possibilities:

(i) **Diffusion dominated case:** $\rho(t)$ decays to zero in some L^p norm with $p > 1$. In this case, the repulsive effects dominates.

(ii) **Aggregation dominated case:** $\rho(t)$ concentrates to a singular measure (delta) in finite or infinite time. Here, the aggregation effect dominates.

(iii) **Balanced case:** $\rho(t)$ converges to some (stable) non trivial L^1 *steady state* for large times.

Unlike the Keller-Segel system, here no mass threshold phenomenon occurs, since the equation is quadratically homogeneous.

A minimization problem

$$\mathrm{argmin}_{\rho\in L^1_+(\mathbb{R}^d)}\left\{\int_{\mathbb{R}^d}\Phi(\rho(x))dx-\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\rho(x)\rho(y)G(x-y)dxdy\right\}.$$

Existence of nontrivial minimizers[5] under the assumptions

- Total mass sufficiently large,
- $\Phi(tu)\leq t^{\nu}\Phi(u)$ with $1<\nu<2$,
- G slow decaying at infinity, i. e. $G(tx)\geq t^{-\alpha}G(x)$ with $\alpha\in(0,d)$,
- $\Phi(u)=o(u^{1+\frac{\alpha}{d}})$ as $u\to 0$.

[5] [Lions - Ann. Inst. H. Poincare 1984]

A critical exponent

Nontrivial minimizers exist[6] if

- $G \in L^1_+$,
- $\Phi(u) = cu^2 + o(u^2)$ as $u \to 0$ with $c > 0$,
- either $c = 0$ or $2c < \int G$.

Case $\Phi(u) = u^m$: the exponent $m = 2$ is *critical*:

- $m > 2 \Rightarrow$ aggregation dominates $\Rightarrow$ nontrivial stationary patterns,
- $m < 2 \Rightarrow$ diffusion dominates (large time decay expected),
- $m = 2 \Rightarrow$??

[6] [Bedrossian, 2012]

Stationary states in multiple dimensions.

$$\frac{\partial \rho}{\partial t} = \mathrm{div}(\rho \nabla(\varepsilon \rho + G * \rho)).$$

Threshold phenomenon[a]

[a] [Burger, DF, Franek - to appear on CMS], [Bedrossian, AML 2011]

- Let $\varepsilon < \|G\|_{L^1}$. Then, there exists at least one non trivial L^1 steady state, which is also a minimizer for the energy $E[\rho]$.
- Let $\varepsilon \geq \|G\|_{L^1}$. Then, there exist no steady states except $\rho \equiv 0$.

Finite time concentration is not possible under the present smoothness assumptions on G.

Stationary points of $E[\rho]$ are steady states and vice-versa

Uniqueness of steady states in one space dimension

With $d = 1$ we can characterize all the steady states as follows.

Theorem (Burger-DF-Franek - to appear on CMS)

Let $\varepsilon < \|G\|_{L^1}$. Then, there exists a unique $\rho \in L^2 \cap \mathcal{P}$ with zero center of mass which solves

$$\rho \partial_x (\varepsilon \rho + G * \rho) = 0.$$

Moreover,

- ρ *is symmetric and monotonically decreasing on* $x > 0$,
- $\rho \in C^2(\mathrm{supp}[\rho])$,
- $\mathrm{supp}[\rho]$ *is a bounded interval in* $\mathbb{R}$,
- ρ *has a global maximum at* $x = 0$ *and* $\rho''(0) < 0$,
- ρ *is the global minimizer of the energy* $E[\rho] = \frac{\varepsilon}{2} \int \rho^2 dx - \frac{1}{2} \int \rho G * \rho dx$.

Sketch of the proof

- Fix $L > 0$. Look for $\rho \in C(\mathbb{R})$ symmetric on $\mathrm{spt}\rho = [-L, L]$, strictly decreasing on $(0, L]$:

$$\varepsilon\rho(x) = -\int_0^L \left(G(x-y) + G(x+y)\right)\rho(y)dy + C \tag{12}$$

- Differentiate (12) w.r.t. x, set $u(x) = -\rho_x(x)$:

$$\varepsilon u(x) = -\int_0^L \left(G(x-y) - G(x+y)\right)u(y)dy =: \mathcal{G}_L[u](x) \tag{13}$$

- Solve the eigenvalue problem (13) with Krein-Rutman theorem. $\mathcal{G}_L$ is a *strictly positive* operator, therefore $\varepsilon = \varepsilon(L)$ is a *simple* eigenvalue $\Rightarrow$ uniqueness of $\rho(x) = \int_x^L u(y)dy$ with $\int_0^L \rho(x)dx = 1$.
- Prove that the function $(0, +\infty) \ni L \mapsto \varepsilon(L) \in (0, 1)$ is continuous and $1:1$ $\Rightarrow$ uniqueness is proven provided all steady states are supported on a bounded interval, symmetric and decreasing on $x > 0$.
- Prove that all steady states are as above. Main tools: symmetric rearrangement and connected support.

Remarks and open problems:

- The uniqueness is surprising because the functional is neither geodesically convex in the Wasserstein space nor convex in the classical sense.
- Uniqueness in many dimensions? We believe it true in the radially symmetric case.
- Porous medium exponent $\gamma \neq 2$ (ongoing discussion with M. Burger, R. Fetecau, Y. Huang).

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The JKO scheme produces entropy solutions

- Nonlocal interaction equations with nonlinear diffusion

$$\partial_t \rho = \Delta \rho^m + \mathrm{div}(\rho \nabla G * \rho) = 0 \tag{14}$$

  with $m > 1$ and $G \in C^2$ and G even. Here, both notions of *entropy solutions* and *gradient flow solutions* have been used (almost at the same time!) to prove uniqueness of solutions.

- Nonlinear diffusion equations with in-homogeneous term

$$\partial_t \rho = \partial_x(\rho \partial_x(a(x)\rho^{m-1}))$$

  with $a(x) \geq c > 0$. In [DF, Matthes - submitted 2012] we prove that the notions of gradient flow solution and entropy solutions coincide.

The results in [DF, Matthes] can be applied also for (14).

A one dimensional repulsive equation[7]

Consider ρ gradient flow solution to

$$\rho_t = \partial_x(\rho\partial_x(G * \rho)), \qquad G(x) = -|x|. \tag{15}$$

Let

$$F(x,t) = \int_{-\infty}^{x} \rho(y,t)dy,$$

then F is an entropy solution to the Burgers' type equation

$$F_t + (F^2 - F)_x = 0. \tag{16}$$

Applications:

- Smoothing effect: initial deltas become densities,
- Wave front tracking approximation for (16) provide particle approximation for (15).

[7] Work in preparation with G. Bonaschi and J. A. Carrillo

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A two species model[8]

- $X_1, \ldots, X_N$ particles of the first species with masses $n_1, \ldots, n_N$,
- $Y_1, \ldots, Y_M$ are particles of the second species with masses $m_1, \ldots, m_M$.

Particle system:

$$\begin{cases} \dot{X}_i(t) = -\sum_{X_i \neq X_k} n_k \nabla H_1(X_i(t) - X_k(t)) - \sum_{X_i \neq Y_k} m_k \nabla K_1(X_i(t) - Y_k(t)) \\ \dot{Y}_j(t) = -\sum_{Y_j \neq Y_k} m_k \nabla H_2(Y_j(t) - Y_k(t)) - \sum_{Y_j \neq X_k} n_k \nabla K_2(Y_j(t) - X_k(t)) \end{cases} .$$

Continuum version:

$$\begin{cases} \partial_t \mu_1 = \operatorname{div}\left(\mu_1 \nabla H_1 * \mu_1 + \mu_1 \nabla K_1 * \mu_2\right) \\ \partial_t \mu_2 = \operatorname{div}\left(\mu_2 \nabla H_2 * \mu_2 + \mu_2 \nabla K_2 * \mu_1\right). \end{cases}$$

[8] [DF, Fagioli - submitted]

Motivation

- Pedestrian movements, lane formation, segregation, cf. [Appert-Rolland, Degond, Motsch - 2011], [Colombo, Lécureux-Mercier - 2012].
- Opinion formation, cf. [Josek - 2009], [Düring, Markowich, Pietschmann, Wolfram - 2009], [Escudero, Macià, Velázquez - 2010].
- Two species chemotaxis, cf. [Horstmann - 2011], [Espejo, Stevens, Velázquez - 2009], [Conca, Espejo, Vilches - 2011].
- Predator–Prey type interaction, cf. [Mogilner, Edelstein-Keshet, Bent, Spiros - 2003].

Symmetrizable case

$$\begin{cases} \partial_t \mu_1 = \mathrm{div}\left(\mu_1 \nabla K_{11} * \mu_1 + \mu_1 \nabla K_{12} * \mu_2\right) \\ \partial_t \mu_2 = \alpha \mathrm{div}\left(\mu_2 \nabla K_{22} * \mu_2 + \mu_2 \nabla K_{12} * \mu_1\right). \end{cases} \tag{17}$$

System (17) has a gradient flow structure, with functional

$$\mathbf{F}(\mu_1, \mu_2) = \frac{1}{2} \int_{\mathbb{R}^d} K_{11} * \mu_1 d\mu_1 + \frac{1}{2} \int_{\mathbb{R}^d} K_{22} * \mu_2 d\mu_2 + \int_{\mathbb{R}^d} K_{12} * \mu_2 d\mu_1 .$$

The quantity

$$c_{M,\alpha} := \alpha \int x d\mu_1(x) + \int x d\mu_2(x)$$

is preserved.

Metric product structure

$$\mu = (\mu_1, \mu_2) \in \mathscr{P}_2(\mathbb{R}^d) \times \mathscr{P}_2(\mathbb{R}^d),$$

$$\mathcal{W}_{2,\alpha}^2(\mu, \nu) = W_2^2(\mu_1, \nu_1) + \frac{1}{\alpha} W_2^2(\mu_2, \nu_2).$$

Results in the symmetrizable case

Assumptions: all the kernels K_{ij} are mildly singular and λ_{ij}–convex. We prove:

- λ convexity of the interaction energy on a suitable sub-differential structure.
- Existence, uniqueness, and stability of gradient flow solutions, by generalizing the one-species theory.
- Finite time collapse if all the kernels are of Non–Osgood type.
- Partial intermediate collapse of each species if the cross interaction kernel decays at infinity.

General case: the strategy

No gradient flow structure in general, no variational formulation. Main idea: semi-implicit version of the JKO scheme.
For all $\mu \in \mathscr{P}(\mathbb{R}^d)^2$ we set

$$\mathbf{F}[\mu|\nu] = \frac{1}{2}\int_{\mathbb{R}^d} H_1 * \mu_1 d\mu_1 + \int_{\mathbb{R}^d} K_1 * \nu_2 d\mu_1 + \frac{1}{2}\int_{\mathbb{R}^d} H_2 * \mu_2 d\mu_2 + \int_{\mathbb{R}^d} K_2 * \nu_1 d\mu_2.$$

Let $\tau > 0$ be a fixed time step, and let $\mu_0 = (\mu_{0,1}, \mu_{0,2}) \in \mathscr{P}(\mathbb{R}^d)^2$ be a fixed initial pair of probability measures. For a given $\mu_n^\tau \in \mathscr{P}(\mathbb{R}^d)^2$, we define the sequence μ_{n+1}^τ as

$$\mu_\tau^{n+1} \in \text{argmin}_{\mu \in \mathscr{P}_2(\mathbb{R}^d)\times\mathscr{P}_2(\mathbb{R}^d)} \left\{ \frac{1}{2\tau}\mathcal{W}_2^2(\mu_\tau^n, \mu) + \mathbf{F}\left[\mu|\mu_\tau^n\right] \right\}.$$

General case: the results

- Existence of weak measure solutions

$$\frac{d}{dt}\int \phi(x)d\mu_1(x,t) = -\frac{1}{2}\iint \nabla H_1(x-y)\cdot(\nabla\phi(x)-\nabla\phi(y))d\mu_1(x)d\mu_1(y)$$
$$-\iint \nabla K_1(x-y)\cdot\nabla\phi(x)d\mu_1(x)d\mu_2(y)$$
$$\frac{d}{dt}\int \psi(x)d\mu_2(x,t) = -\frac{1}{2}\iint \nabla H_2(x-y)\cdot(\nabla\psi(x)-\nabla\psi(y))d\mu_2(x)d\mu_2(y)$$
$$-\iint \nabla K_2(x-y)\cdot\psi(x)d\mu_2(x)d\mu_1(y).$$

  as limit of the semi-implicit JKO scheme.

- Uniqueness in case H_j and K_j are $W^{2,\infty}$, via a variant of the characteristics method.

Open problems and future work

- Open problem: uniqueness in the two species system for less regular potentials.
- Many species with nonlocal aggregation and nonlinear cross-diffusion terms: segregation. Ongoing project with M. Burger and A. Stevens.
- Derivation of multi-species continuum second order models via particle methods.
- Derivation of first order systems as damping dominated limits of second order systems.

End of the talk

Thank you for your attention!