Different types of phase transitions for a simple model of alignment of oriented particles

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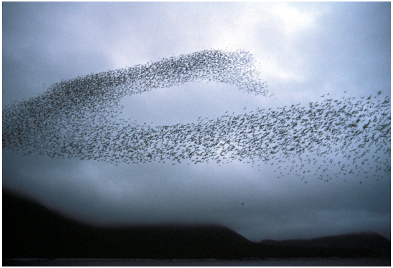
Goal: macroscopic description of some animal societies



- Local interactions without leader
- Emergence of macroscopic structures, phase transitions

Modeling of interacting self-propelled particles

- Vicsek *et al.* (1995).
  Discrete in time (interval Δt), alignment only, synchronous reorientation.

  $$\text{New direction} = \text{Mean direction of neighboring particles at previous step} + \text{Noise}$$

  Simulations: phase transition phenomenon, emergence of coherent structures.
- Degond-Motsch (2008).
  Time-continuous version: relaxation (with constant rate ν) towards the local mean direction.
  Hydrodynamic limit without phase transition phenomenon.
- Model presented here: making ν (and the intensity of the noise) a function of the (norm of the) local mean momentum.

Outline

1. Time-continuous Vicsek model with phase transition
   - Presentation of the model
   - Kinetic model – Homogeneous setting
   - Stability issues
2. Examples of different types of phase transitions
   - Second order (or continuous) phase transition
   - First order (discontinuous) phase transition – Hysteresis

Individual dynamics

Particles at positions: $X_1, \dots, X_N$ in $\mathbb{R}^n$.
Orientations $\omega_1, \dots, \omega_N$ in $\mathbb{S}$ (unit sphere).

$$\begin{cases} \mathrm{d}X_k = \omega_k \mathrm{d}t \\ \mathrm{d}\omega_k = \nu P_{\omega_k^\perp} \bar{\omega}_k \mathrm{d}t + \sqrt{2\tau}\, P_{\omega_k^\perp} \circ \mathrm{d}B_t^k \end{cases}$$

Target direction:

$$\bar{\omega}_k = \frac{J_k}{|J_k|}, \quad J_k = \frac{1}{N} \sum_{j=1}^{N} K(X_j - X_k)\, \omega_j.$$

Setting $\nu = \nu(|J_k|)$ and $\tau = \tau(|J_k|)$, no singularity if $\frac{\nu(|J|)}{|J|}$ is Lipschitz.

Kinetic description

Assumptions : K with finite second moment, and K, $|J| \mapsto \frac{\nu(|J|)}{|J|}$ and τ bounded Lipschitz.

Theorem (following Bolley, Cañizo, Carrillo, 2012)

Probability density function $f(x,\omega,t)$, as $N \to \infty$:

$$\partial_t f + \omega \cdot \nabla_x f + \nu(|\mathcal{J}_f|)\nabla_\omega \cdot (P_{\omega^\perp}\bar{\omega}_f\, f) = \tau(|\mathcal{J}_f|)\Delta_\omega f$$

$$\bar{\omega}_f = \frac{\mathcal{J}_f}{|\mathcal{J}_f|}, \quad \mathcal{J}_f = K * J_f, \quad J_f = \int_{\omega\in\mathbb{S}} \omega f(x,\omega,t)\,\mathrm{d}\omega\,.$$

Tool : coupling process + estimations.

$$\begin{cases} \mathrm{d}\bar{X}_k = \bar{\omega}_k \mathrm{d}t \\ \mathrm{d}\bar{\omega}_k = \nu(|\mathcal{J}_{f_t^N}|)P_{\omega_k^\perp}\,\bar{\omega}_{f_t^N}\,\mathrm{d}t + \sqrt{2\tau(|\mathcal{J}_{f_t^N}|)}P_{\omega_k^\perp}\circ \mathrm{d}B_t^k \\ f_t^N = \mathrm{law}(\bar{X}_1,\bar{\omega}_1) = \mathrm{law}(\bar{X}_k,\bar{\omega}_k) \end{cases}$$

Space-homogeneous version

Reduced equation, for a function $f(\omega, t)$:

$$\partial_t f = Q(f),$$
$$Q(f) = -\nu(|J_f|)\nabla_\omega \cdot (P_{\omega^\perp}\Omega_f f) + \tau(|J_f|)\Delta_\omega f,$$
$$\Omega_f = \frac{J_f}{|J_f|}, \quad J_f(t) = \int_{\mathbb{S}} f(\omega, t)\, \omega \, \mathrm{d}\omega.$$

Key parameter: the conserved quantity $\rho = \int_{\mathbb{S}} f$.
Writing $h(|J|) = \frac{\nu(|J|)}{\tau(|J|)}$, we get

$$Q(f) = \tau(|J_f|)\nabla_\omega \cdot (e^{h(|J_f|)\omega\cdot\Omega_f}\nabla_\omega(e^{-h(|J_f|)\omega\cdot\Omega_f} f)).$$

Main assumption: $|J| \mapsto h(|J|)$ is increasing. Its inverse: σ.

Equilibria

Definitions: Fisher–von Mises distribution

$$M_{\kappa\Omega}(\omega) = \frac{e^{\kappa\,\omega\cdot\Omega}}{\int_{\mathbb{S}} e^{\kappa\,\upsilon\cdot\Omega}\,\mathrm{d}\upsilon}.$$

Orientation $\Omega \in \mathbb{S}$, concentration $\kappa \geqslant 0$.

Order parameter: $c(\kappa) = |J_{M_{\kappa\Omega}}| = \frac{\int_0^\pi \cos\theta\, e^{\kappa\cos\theta}\sin^{n-2}\theta\,\mathrm{d}\theta}{\int_0^\pi e^{\kappa\cos\theta}\sin^{n-2}\theta\,\mathrm{d}\theta}$.

For $\kappa_f = h(|J_f|)$, we can write Q under the form:

$$Q(f) = \tau(|J_f|)\nabla_\omega\cdot\left[M_{\kappa_f\Omega_f}\nabla_\omega\left(\frac{f}{M_{\kappa_f\Omega_f}}\right)\right].$$

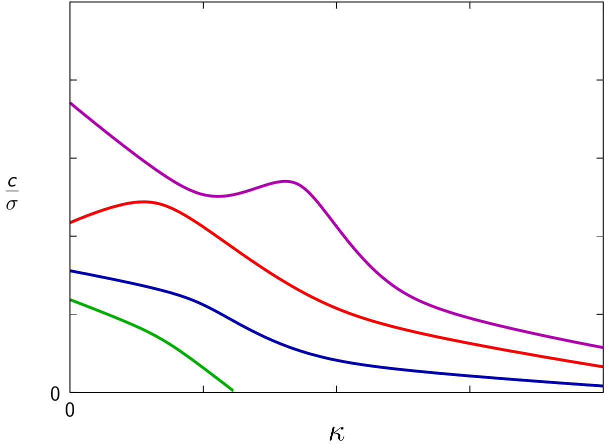
Equilibria: $f_{eq} = \rho M_{\kappa\Omega}$, for some $\Omega \in \mathbb{S}$.

Then $|J_{f_{eq}}| = \rho|J_{M_{\kappa\Omega}}| = \rho c(\kappa)$, and $\kappa = \kappa_{f_{eq}} = h(|J_{f_{eq}}|)$.

Compatibility condition: $\kappa = h(\rho c(\kappa))$, i.e. $\sigma(\kappa) = \rho c(\kappa)$.

Solutions to the compatibility condition

Uniform distribution $f = \rho$: always equilibrium.
Possible shapes for function $\frac{c}{\sigma}(=\frac{1}{\rho})$? 2 informations: $\to \frac{1}{\rho_c}$ as $\kappa \to 0$, and $\to 0$ as $\kappa \to \kappa_{max}$ (the maximum of $h(|J|)$).



Existence, uniqueness, regularity, positivity, bounds

Theorem

For an initial probability measure $f_0 \in H^s(\mathbb{S})$, (for an arbitrary s):

- Existence and uniqueness of a weak solution f.
- Global solution, in $C^\infty(\mathbb{R}_+^* \times \mathbb{S})$, and $f > 0$ for $t > 0$.
- Instantaneous regularity estimates and uniform bounds:

$$\|f(t)\|^2_{H^{s+m}} \leqslant C\left(1 + \frac{1}{t^m}\right)\|f_0\|^2_{H^s}.$$

Tool: spherical harmonics decomposition.
Nonlinearity: finite number of coefficients.

Main tool: Onsager free energy

Free energy: $\mathcal{F}(f) = \int_{\mathbb{S}} f \ln f - \Phi(|J_f|)$, with $\frac{\mathrm{d}\Phi}{\mathrm{d}|J|} = h(|J|)$.
Dissipation: $\mathcal{D}(f) = \tau(|J_f|) \int_{\mathbb{S}} f |\nabla_\omega(\ln f - h(|J_f|)\omega \cdot \Omega_f)|^2 \geqslant 0$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} + \mathcal{D} = 0 \quad \Rightarrow \mathcal{F}(f) \text{ decreasing towards } \mathcal{F}_\infty.$$

LaSalle's principle

Limit set: $\mathcal{E}_\infty = \{f \in C^\infty(\mathbb{S}) | \mathcal{D}(f) = 0 \text{ and } \mathcal{F}(f) = \mathcal{F}_\infty\}$.

$$\lim_{t\to\infty} \inf_{g\in\mathcal{E}_\infty} \|f(t) - g\|_{H^s} = 0.$$

Refined: if the roots of $\sigma(\kappa) = \rho c(\kappa)$ are isolated, there exists a solution κ_∞ such that:

$$\lim_{t\to\infty} |J_f(t)| = \rho c(\kappa_\infty) \quad \text{and} \quad \forall s \in \mathbb{R}, \lim_{t\to\infty} \|f(t) - \rho M_{\kappa_\infty \Omega_f(t)}\|_{H^s} = 0.$$

Local analysis near uniform equilibria

Critical value: $\rho_c = \lim_{\kappa \to 0} \frac{\sigma(\kappa)}{c(\kappa)} \in (0, +\infty]$.

Theorem: Strong unstability – Exponential stability

- $\rho > \rho_c$: if $J_{f_0} \neq 0$, then f cannot converge to the uniform equilibrium.
- $\rho < \rho_c$: There exists a universal constant δ such that if $\|f_0 - \rho\|_{H^s} < \delta$, then we have for all $t \geqslant 0$

$$\|f(t) - \rho\|_{H^s} \leqslant \frac{\|f_0 - \rho\|_{H^s} e^{-\lambda t}}{1 - \frac{1}{\delta}\|f_0 - \rho\|_{H^s}}, \text{ with } \lambda = (n-1)\tau_0(1 - \frac{\rho}{\rho_c}).$$

Tools: linearization for the evolution of J_f, and then energy estimates for the whole equation.

Around a nonisotropic equilibria $\rho M_{\kappa\Omega}$

Proposition: weak stability/unstability

We denote by $\mathcal{F}_\kappa$ the value of $\mathcal{F}(\rho M_{\kappa\Omega})$ (independent of $\Omega \in \mathbb{S}$).

- $(\frac{\sigma}{c})'(\kappa) < 0$ (unstable): in any neighborhood of $\rho M_{\kappa\Omega}$, there exists f_0 such that $\mathcal{F}(f_0) < \mathcal{F}_\kappa$.
- $(\frac{\sigma}{c})'(\kappa) > 0$ (stable): If $\|f_0\|_{H^s} \leqslant K$, with $s > \frac{n-1}{2}$, then there exists $\delta > 0$ and C (depending only on K and s), such that if $\|f_0 - \rho M_{\kappa\Omega}\|_{L^2} \leqslant \delta$ for some $\Omega \in \mathbb{S}$, then for all $t \geqslant 0$, we have

$$\mathcal{F}(f) \geqslant \mathcal{F}_\kappa, \text{ and } \|f - \rho M_{\kappa\Omega_f}\|_{L^2} \leqslant C\|f_0 - \rho M_{\kappa\Omega_{f_0}}\|_{L^2}.$$

Tools: expansion of $\mathcal{F} - \mathcal{F}_\kappa$, and Sobolev interpolation.

Stronger stability: exponential convergence to equilibrium

Theorem: exponential stability in case $(\frac{\sigma}{c})'(\kappa) > 0$

For all $s > \frac{n-1}{2}$, there exist universal constants $\delta > 0$ and $C > 0$, such that if $\|f_0 - \rho M_{\kappa\Omega}\|_{H^s} < \delta$ for some $\Omega \in \mathbb{S}$, there exists $\Omega_\infty \in \mathbb{S}$ such that

$$\|f - \rho M_{\kappa\Omega_\infty}\|_{H^s} \leqslant C\|f_0 - \rho M_{\kappa\Omega_{f_0}}\|_{H^s} e^{-\lambda t},$$

with

$$\lambda = \frac{c\tau(\sigma)}{\sigma'}\Lambda_\kappa(\frac{\sigma}{c})',$$

where Λ_κ is the best constant for the following weighted Poincaré inequality:

$$\langle|\nabla g|^2\rangle_{M_{\kappa\Omega}} \geqslant \Lambda_\kappa\langle(g - \langle g\rangle_{M_{\kappa\Omega}})^2\rangle_{M_{\kappa\Omega}}$$

Outline

General statements in the case $(\frac{\sigma}{c})' > 0$ for all κ

- If $\rho < \rho_c$, then the solution converges exponentially fast towards the uniform distribution $f_\infty = \rho$.
- If $\rho = \rho_c$, the solution converges to the uniform distribution.
- If $\rho > \rho_c$ and $J_{f_0} \neq 0$, then there exists Ω_∞ such that f converges exponentialy fast to the von Mises distribution $f_\infty = \rho M_{\kappa\Omega_\infty}$, where $\kappa > 0$ is the unique positive solution to the equation $\rho c(\kappa) = \sigma(\kappa)$.

We can then define c (order parameter) as a function of ρ, and this function is continuous.
Critical exponent β: when $c(\rho) \asymp (\rho - \rho_c)^\beta$. Can be any number in $(0, 1]$, as one can artificially choose $\sigma(\kappa) = c(\kappa)(1 + \kappa^{\frac{1}{\beta}})$.

Practical criteria for continuous phase transition

Lemma

If $\frac{h(|J|)}{|J|}$ is a nonincreasing function of $|J|$, then we are in the previous case of a continuous phase transition. In that case, the critical exponent β, if it exists, can only take values in $[\frac{1}{2}, 1]$.

In that case, we also have a special cancellation (related to the so-called conformal Laplacian), which gives global exponential decay when $\rho < \rho_c$:

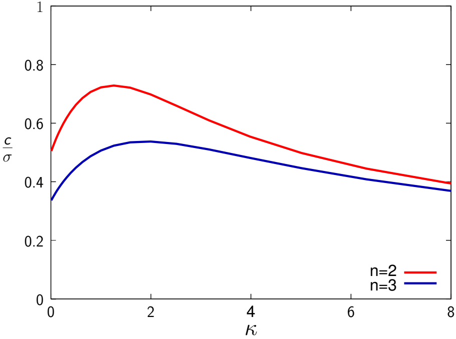
Proposition

If $\rho < \rho_c$, there exists a universal constant C such that we have for all $t \geqslant 0$

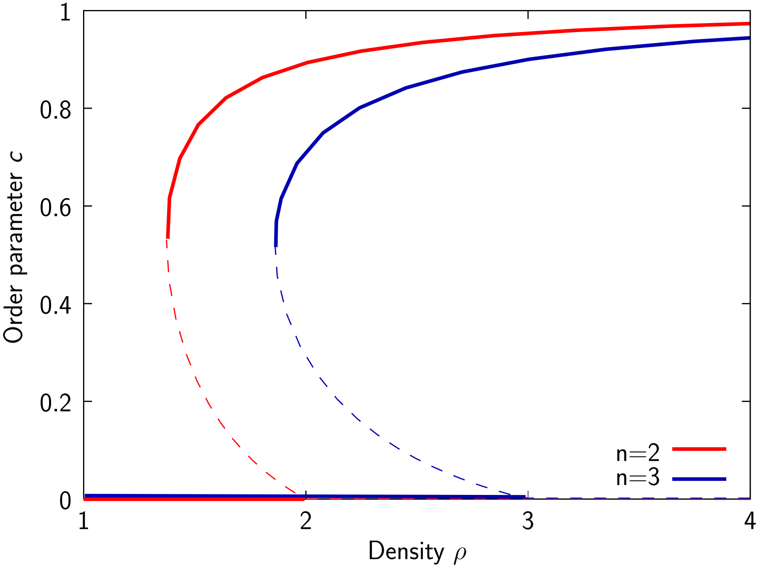
$$\|f(t) - \rho\|_{H^s} \leqslant C\|f_0 - \rho\|_{H^s} e^{-\lambda t}, \text{ with } \lambda = (n-1)\tau_{min}(1 - \frac{\rho}{\rho_c}).$$

A special example

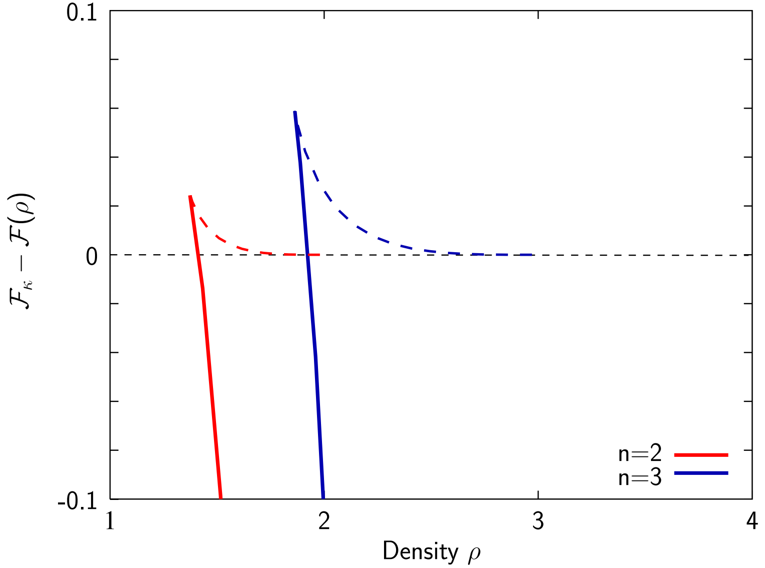
Focus: $\nu(|J|) = |J|$ and $\tau(|J|) = \dfrac{1}{1+|J|}$ (related to the so-called "extrinsic noise").
In that case, we get $h(|J|) = |J| + |J|^2$, so we are not indeed in the previous case.
We obtain $\sigma(\kappa) = \frac{1}{2}(\sqrt{1+4\kappa} - 1)$.



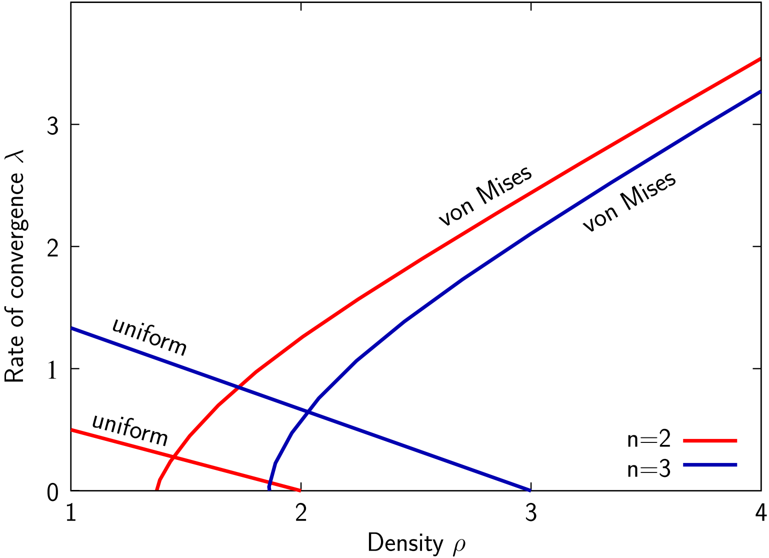
The phase diagram



Comparison of Free energy levels



Rates of convergence



Numerical illustration of the hysteresis phenomena

Change of scale $\widetilde{f} = \frac{f}{\rho}$. The parameter ρ can now be considered as a free parameter that we let evolve in time.