An asymptotic-preserving scheme for linear kinetic equation with fractional diffusion limit

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Joint with **Bokai Yan**

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Outline

1. Linear Boltzmann equation and fractional diffusion limit
2. Asymptotic preserving scheme
3. Numerical examples
4. Conclusion

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Linear Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f = \mathcal{L}(f), \qquad (t, x, v) \in (0, \infty) \times \mathbf{R}^N \times \mathbf{R}^N,$$
$$f(0, x, v) = f_0(x, v),$$

where the collision $\mathcal{L}$ takes the form

$$\mathcal{L}(f) = \int_{\mathbf{R}^N} [\sigma(x, v, v') f(t, x, v') - \sigma(x, v', v) f(t, x, v)] \, \mathrm{d}v'.$$

- $\sigma(x, v, v') \geq 0$ is the transition probability
- $\mathcal{L}$ has a unique equilibrium function $\mathcal{F}(v) \geq 0$ satisfying

$$\mathcal{L}(\mathcal{F}) = 0, \qquad \mathcal{F}(v) = \mathcal{F}(-v), \qquad \int_{\mathbf{R}^N} \mathcal{F}(v) \, \mathrm{d}v = 1 \quad \text{for all} \quad x \in \mathbf{R}^N.$$

Classical diffusion limit

ϵ: ratio of the mean free path over the macroscopic length scale.
$x' = \epsilon x$, $t' = \epsilon^2 t$:

$$\epsilon^2 \partial_t f + \epsilon v \cdot \nabla_x f = \mathcal{L}(f).$$

Hilbert expansion: $f = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \cdots$

$$O(1): \ \mathcal{L}(f_0) = 0 \quad \Longrightarrow \ f_0 = \rho(t,x)\mathcal{F}(v)$$
$$O(\epsilon): \ v \cdot \nabla_x f_0 = \mathcal{L}(f_1) \quad \Longrightarrow \ f_1 = \mathcal{L}^{-1}(v \cdot \nabla_x f_0)$$
$$O(\epsilon^2): \ \partial_t f_0 + v \cdot \nabla_x f_1 = \mathcal{L}(f_2) \quad \Longrightarrow \ \partial_t \rho - \nabla_x (D \nabla_x \rho) = 0$$
$$D = \int_{R^n} v \otimes \mathcal{L}^{-1}(v\mathcal{F})\, \mathrm{d}v$$

Fractional diffusion limit

Consider

$$\mathcal{F}(v) \sim \frac{\kappa_0}{|v|^{N+\alpha}}, \qquad 1 < \alpha < 2, \qquad \text{as } |v| \to \infty.$$

Applications:

- granular plasma with dissipative collision
- astrophysical plasma
- economy
- $\cdots$

$D = \int_{R^n} v \otimes \mathcal{L}^{-1}(v\mathcal{F})\, dv$ is infinite. Classical diffusion theory fails:(

We consider a different scaling:

$$\epsilon^{\alpha}\partial_t f + \epsilon v \cdot \nabla_x f = \mathcal{L}(f).$$

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Fractional diffusion limit: simple case

$$\epsilon^\alpha \partial_t f + \epsilon v \cdot \nabla_x f = \langle f \rangle \mathcal{F} - f, \qquad 1 < \alpha < 2$$

Insert Hilbert expansion $f = f_0 + g_1 + g_2 + \cdots$, then the leading terms solve [1]

$$0 = \mathcal{L}(f_0) = \langle f_0 \rangle \mathcal{F} - f_0$$

$$\Longrightarrow f_0 = \rho_0 \mathcal{F}(v)$$

$$\epsilon v \cdot \nabla_x (f_0 + g_1) = -g_1$$

$$\Longrightarrow (1 + i\epsilon v \cdot k)\hat{g_1} = -i\epsilon v \cdot k \hat{f_0}$$

$$\frac{1}{\epsilon^\alpha} \langle \hat{g_1} \rangle = \frac{1}{\epsilon^\alpha} \int_{\mathbf{R}^N} \frac{i\epsilon v \cdot k}{1 + i\epsilon v \cdot k} \mathcal{F}(v) \, \mathrm{d}v \hat{\rho}_0 \to \kappa |k|^\alpha \hat{\rho}_0$$

$$\epsilon^\alpha \partial_t f_0 = \langle g_1 \rangle \mathcal{F} + \mathcal{L}(g_2)$$

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Fractional diffusion limit: general case

$$\epsilon^\alpha \partial_t f + \epsilon v \cdot \nabla_x f = K(f)\mathcal{F} - \nu(x,v) f$$

$$K(f) = \int_{\mathbf{R}^N} \phi(x,v,v') f(t,x,v')\,\mathrm{d}v', \qquad \nu(x,v) = \int_{\mathbf{R}^N} \phi(x,v',v)\mathcal{F}(v')\,\mathrm{d}v'$$

$\epsilon \to 0$

$$\partial_t \rho_0 + L(\rho_0) = 0$$

$$L(\rho) = \kappa_0 P.V. \int_{\mathbf{R}^N} \gamma(x,y) \frac{\rho(x)-\rho(y)}{|y-x|^{N+\alpha}}\,\mathrm{d}y$$

$$\gamma(x,y) = \nu_0(x)\nu_0(y) \int_0^\infty z^\alpha e^{-z\int_0^1 \nu_0((1-s)x+sy)ds}\,\mathrm{d}z$$

- we use an integral formulation of $\nu(x) + \epsilon v \cdot \nabla_x$
- when $\phi(x,v,v') = 1$, $\nu(v) \equiv 1$ and thus $\nu_0 = 1$. Then $L(\rho) = \kappa_0 P.V. \int_{\mathbf{R}^N} \frac{\rho(x)-\rho(y)}{|y-x|^{N+\alpha}}\,\mathrm{d}y$, which is the integral representation of $\kappa(-\triangle)^{\frac{\alpha}{2}}$.

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Numerical issue

Our goal: design a scheme uniformly stable in both kinetic ($\epsilon = O(1)$) and diffusive ($\epsilon \ll 1$) regimes.

Difficulties:

1. stiffness
2. [2] *reshuffled* Hilbert expansion
3. *fat tail* in the equilibrium

$$\epsilon^{\alpha}\partial_t f + \epsilon v \cdot \nabla_x f = \langle f \rangle \mathcal{F} - f, \qquad 1 < \alpha < 2$$

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System decomposition

$$\begin{cases} \epsilon^\alpha \partial_t f + \epsilon v \cdot \nabla_x f = \langle f \rangle \mathcal{F} - f, & 1 < \alpha < 2, \\ f(0,x) = f_{\text{in}}(x). \end{cases}$$

1. Decompose $f = \rho\mathcal{F} + g$, then we have
$\epsilon^\alpha \partial_t(\rho\mathcal{F} + g) + \epsilon v \cdot \nabla_x(\rho\mathcal{F} + g) = \langle \rho\mathcal{F} + g \rangle \mathcal{F} - (\rho\mathcal{F} + g)$;

2. Split the system into two sub-equations

$$\begin{aligned} \epsilon^\alpha \partial_t \rho &= \langle g \rangle, \\ \epsilon^\alpha \partial_t g + \epsilon v \cdot \nabla_x(\rho\mathcal{F} + g) &= -g; \end{aligned}$$

Notice: $\langle g \rangle \neq 0$ for finite ϵ.

3. This splitting is well-posed.

4. Initial data decomposition

$$\begin{cases} \rho_{\text{in}} = \langle f_{\text{in}} \rangle - \langle g_{\text{in}} \rangle = \langle f_{\text{in}} + \epsilon v \cdot \nabla_x f_{\text{in}} \rangle, \\ g_{\text{in}} = f_{\text{in}} - \rho_{\text{in}}\mathcal{F}. \end{cases}$$

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First order semi-discrete scheme: preliminary version

$$\begin{cases} \epsilon^\alpha \dfrac{\rho^{n+1}-\rho^n}{\Delta t} = \langle g^{n+1} \rangle, \\ \epsilon^\alpha \dfrac{g^{n+1}-g^n}{\Delta t} + \epsilon v \cdot \nabla_x (\rho^* \mathcal{F} + g^{n+1}) = -g^{n+1}, \qquad \rho^* = \rho^{n+1} \text{ or } \rho^n. \end{cases}$$

- AP property:

$$\epsilon^\alpha \frac{\hat{g}^{n+1}-\hat{g}^n}{\Delta t} + i\epsilon v \cdot k (\hat{\rho}^* \mathcal{F} + \hat{g}^{n+1}) = -\hat{g}^{n+1}$$

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- Stability: unconditionally stable for $\rho^* = \rho^{n+1}$.

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Velocity truncation?

Recall

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then a truncation of v in $|v| < v_{\max}$ brings in an error

$$\frac{1}{\epsilon^\alpha} \int_{|v|>v_{\max}} \frac{(\epsilon v \cdot k)^2}{1 + (\epsilon v \cdot k)^2} \frac{\kappa_0}{|v|^{N+\alpha}} \, \mathrm{d}v \sim \frac{\kappa_0 S^{N-1}}{\alpha} \frac{1}{v_{\max}^\alpha} |k|^\alpha,$$

which implies that in order to to suppress the error to $O(\epsilon)$, $v_{\max}$ should be chosen as $O(\epsilon^{-\frac{1}{\alpha}})$, not AP!

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Tail compensation

1. *Further decompose* $\mathcal{F}$ and g into

$$\mathcal{F}(v) = \mathcal{F}_B(v) + \mathcal{F}_T(v), \quad g(t,x,v) = g_B(t,x,v) + g_T(t,x,v)$$

$$\mathcal{F}_B(v) = \mathcal{F}(v)\mathbf{1}_{|v|\le v_{\max}}, \qquad \mathcal{F}_T(v) = \mathcal{F}(v)\mathbf{1}_{|v|> v_{\max}},$$
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2. Rewrite the split system as

$$\begin{aligned} \epsilon^\alpha \partial_t \rho &= \langle g_B + g_T \rangle, \\ \epsilon^\alpha \partial_t g_B + \epsilon v\cdot\nabla_x(\rho\mathcal{F}_B + g_B) &= -g_B, \\ \epsilon^\alpha \partial_t g_T + \epsilon v\cdot\nabla_x(\rho\mathcal{F}_T + g_T) &= -g_T. \end{aligned}$$

3. Use an *integrated* tail: approximate g_T by its steady state.

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Choice of $v_{\max}$

Approximating g_T by its steady state is valid when ϵ is small.

We do not want to introduce to much error in $\epsilon^\alpha \partial_t \rho = \langle g_B + g_T \rangle$ for $\epsilon \sim O(1)$.

Need $\frac{1}{\epsilon^\alpha} \langle g_T \rangle \sim O(\triangle t)$ when $\epsilon \sim O(1)$.

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step1 Compute $\frac{1}{\epsilon^\alpha}\left\langle \hat{g}_{_T}^{n+1}(k) \right\rangle$ via $\frac{1}{\epsilon^\alpha}\left\langle \hat{g}_{_T}^{n+1}(k) \right\rangle = C(k)\hat{\rho}^*(k)$.

step2 Solve $\hat{g}_{_B}^{n+1}(k,v)$ for $|v| \leq v_{\max}$ from g_B equation.

step3 Compute $\left\langle \hat{g}_{_B}^{n+1} \right\rangle$ by a simple summation in velocity space.

step4 Plug $\frac{1}{\epsilon^\alpha}\left\langle \hat{g}_{_B}^{n+1} \right\rangle$ and $\frac{1}{\epsilon^\alpha}\left\langle \hat{g}_{_T}^{n+1} \right\rangle$ into $\hat{\rho}^{n+1}$ equation to get $\hat{\rho}^{n+1}(k)$.

Repeat the above three steps until the end of time $t = t^M$, and f^M is recovered from

$$
\hat{f}^M(k,v) = \hat{\rho}^M(k)\mathcal{F}(v) + \hat{g}_{_B}^M(k,v), \qquad \text{for } |v| \leq v_{\max}.
$$

Second order scheme

$$\begin{cases} \epsilon^\alpha \dfrac{3\rho^{n+1} - 4\rho^n + \rho^{n-1}}{2\Delta t} = \left\langle g_{_B}^{n+1} \right\rangle + \left\langle g_{_T}^{n+1} \right\rangle, \\ \epsilon^\alpha \dfrac{3g_{_B}^{n+1} - 4g_{_B}^n + g_{_B}^{n-1}}{2\Delta t} + \epsilon v \cdot \nabla_x (\rho^* \mathcal{F}_{_B} + g_{_B}^{n+1}) = -g_{_B}^{n+1}, \\ \left\langle \hat{g}_{_T}^{n+1} \right\rangle = \displaystyle\int_{|v| \geq v_{\max}} \frac{i\epsilon v \cdot k}{1 + i\epsilon v \cdot k} \mathcal{F}_{_T}(v)\, \mathrm{d}v \hat{\rho}^*. \end{cases}$$

- $\rho^* = 2\rho^n - \rho^{n-1}$ for an explicit scheme and $\rho^* = \rho^{n+1}$ for an implicit scheme.
- $\delta = \triangle t^2$ in the truncation.

Outline

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2. Asymptotic preserving scheme
3. Numerical examples
4. Conclusion

Tail effect

$$I_1(k) := \lim_{\epsilon\to 0} \frac{1}{\epsilon^\alpha}\int_{|v|<v_{\max}} \frac{i\epsilon v\cdot k}{1+i\epsilon v\cdot k}\mathcal{F}(v)\,\mathrm{d}v \quad \left(\lim_{\epsilon\to 0}\frac{1}{\epsilon^\alpha}\int \frac{i\epsilon v\cdot k}{1+i\epsilon v\cdot k}\mathcal{F}(v)\,\mathrm{d}v \to \kappa|k|^\alpha\right)$$

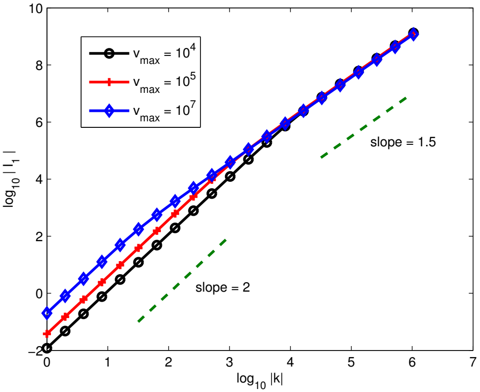

Figure : The integral I_1 as a function of the frequency k, with different cutoff $v_{\max}$ in velocity space. $\alpha = 1.5$.

Uniform convergence

$$\rho(x) = e^{-5(x-\pi)^2}, \quad f(x) = \frac{\rho(x)}{\sqrt{2\pi}} e^{-|v|^2/2}, \quad x \in [0, 2\pi].$$

$$N_x = 100; \quad N_t = 200 \sim 3200; \quad v_{\max} = 40.$$

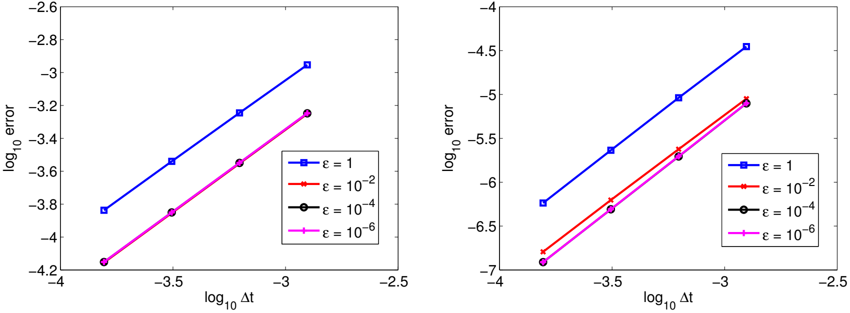

Figure : Convergence test for the first order (left) and second order (right) explicit AP schemes for different ϵ.

Stability

- Dissipation of 'energy' $E_f = \left(\int\int f^2 \mathcal{F}^{-1} dx dv\right)^{1/2}$
- Boundedness of energy $E_\rho = \left(\int \rho^2 dx\right)^{1/2}, \quad E_g = \left(\int\int \frac{g^2}{\mathcal{F}} dx dv\right)^{1/2}$

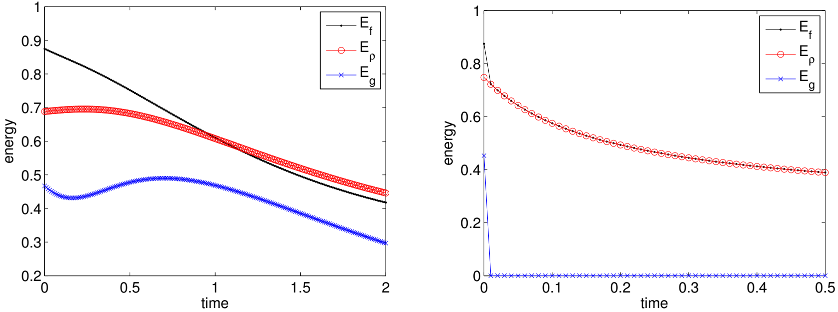

Figure : Left $\epsilon = 1$. Right $\epsilon = 10^{-6}$.

AP property

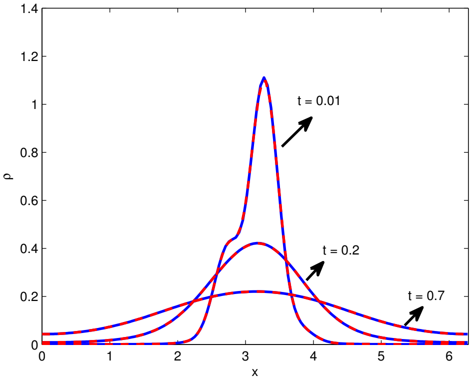

Figure : The snapshots of the density ρ obtained from AP scheme with $\epsilon = 10^{-8}$ (blue solid line) and that by solving the limit equation (red dashed line), at various time. $N_x = 200$ points are used. $\alpha = 1.5$.

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Conclusion

We have construted an AP scheme for the linear Boltzmann equation with fractional diffusion limit.

— uniformly stable for a wide range of ϵ

— implicit terms are treated efficiently

Key ideas:

- A new macro-micro decomposition
- Velocity truncation and tail compensation

Thank you!