QUANTITATIVE ESTIMATES FOR ADVECTIVE EQUATION WITH DEGENERATE ANELASTIC CONSTRAINT

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Abstract. In these proceedings we are interested in quantitative estimates for advective equations with an anelastic constraint in presence of vacuum. More precisely, we derive a quantitative stability estimate and obtain the existence of renormalized solutions. Our main objective is to show the flexibility of the method introduced recently by the authors for the compressible Navier-Stokes' system. This method seems to be well adapted in general to provide regularity estimates on the density of compressible transport equations with possible vacuum state and low regularity of the transport velocity field; the advective equation with degenerate anelastic constraint considered here is another good example of that. As a final application we obtain the existence of global renormalized solution to the so-called lake equation with possibly vanishing topography.

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1. Introduction

New mathematical tools allowing to encode quantitative regularity estimates for the continuity equation written in Eulerian form have been recently developed by the authors [see [11] and [12]] to answer two longstanding problems: Global existence of weak solutions for compressible Navier–Stokes with thermodynamically unstable pressure or with anisotropic viscous stress tensor. These articles provide a new point of view regarding the weak stability procedure (and more precisely on the space compactness for the density) in compressible fluid mechanics compared to what was developed mainly by P.–L. Lions and E. Feireisl *et al.*: See for example [20], [21], [28].

In the present work, we want to show the flexibility of the method introduced in [11, 12] by focusing on quantitative stability estimates for advective equations with a vector field satisfying a degenerate anelastic constraint (linked to a non-negative scalar function). The method itself introduces weights which solve a dual equation and allow to propagate appropriately weighted norms on the initial solution. In a second time, a control on where those weights may vanish allow to deduce global and precise quantitative regularity estimates. For a more general introduction to the method, we refer interested readers to [10].

The theory of existence and uniqueness for advection equations with rough force fields is now quite extensive, and we refer among others to the seminal articles [18], [1], and to [16, 2] for a general introduction to the topic. But quantitative regularity estimates were first derived on the Lagrangian formulation by G. Crippa and C. De Lellis in [15]. The main idea is to identity the "good" trajectories where the flow has some regularity and then proving that those good trajectories have a large probability, which strongly inspired the Eulerian approach that we present here. This type of Lagrangian estimate is also used for example in [5], [6], [23] and [13]. Note that quantitative regularity estimates for nonlinear continuity equations at the Eulerian level have also been introduced in [3], [4] using a nonlocal characterization of compactness in the spirit of [7]. PDE's with anelastic constraints are found in many different settings and we briefly refer for instance to [24], [29], [19], [35], [30], [22] in meteorology, to [8], [25], and to [27] for lakes and [33], to [26] for the dynamics of congestion or floating structures, to [17] for astrophysics and to [14] for asymptotic regime of strong electric fields to understand the importance to study PDEs with anelastic constraints especially the advective equation. As an application, we derive a new existence result for the so-called lake equation with possibly vanishing bathymetry which could vanish. The fact that we can obtain renormalized solutions in the vorticity formulation is in particular a significant improvement compared to previous results such as in [25].

Let us now present more specifically the problem that we consider: Let Ω be a bounded smooth domain in $\mathbb{R}^d$ with $d = 1, 2$ or 3. We study the following advective equation

$$a\,(\partial_t\phi + u\cdot\nabla\phi) = 0 \text{ in } (0,T)\times\Omega \tag{1}$$

with a velocity field u such that

$$\mathrm{div}(au) = 0 \text{ in } (0,T)\times\Omega, \qquad a\,u\cdot n|_{(0,T)\times\partial\Omega} = 0. \tag{2}$$

where a is a given non-negative scalar function which depends only on the space variable and is continuous on $\overline{\Omega}$. The initial condition is given by

$$a\phi|_{t=0} = m_0 \text{ in } \Omega. \tag{3}$$

To avoid assuming any regularity on a, we still need to impose additional conditions on a: There exists a measurable non-negative function $\alpha(x)$, $r > 1$ and $q > p^*$ (with as usual $1/p^* + 1/p = 1$) s.t.

$$\alpha(x) \le a(x),\ \ A(\alpha,a) = \int_\Omega \Big(|\nabla\alpha^{1/p^*}(x)|^q + a(x)\,(|\log\alpha(x)| + |\nabla\log\alpha(x)|^r)\Big)\,dx < \infty. \tag{4}$$

Of course if $a \in W^{1,p}$ with $p > 1$ and $a\,|\log a| \in L^1$ then we could just choose $\alpha = a^k$ with $k \ge 1$. But (4) is far more general as in particular it does not require any regularity on a away from its vanishing set.

An example. To illustrate the condition (4), assume that there exists a Lipschitz domain $O \subset \Omega$ s.t. $a = 0$ on O^c and on O for some exponents $k,\ l > 0$

$$C^{-1}\,\min((d(x,\partial O))^k, 1) \le a(x) \le C\,\min((d(x,\partial O))^l, 1).$$

Then by taking $\alpha = \min((d(x,\partial O))^\theta, 1)$ with $\theta > p^*$, we immediately satisfy (4).

Let us now consider a velocity field u such that (with a slight abuse of notation as $\|u\|_a$ is not a norm)

$$\|u\|_a := \|u\|_{L^\infty_t L^p_a} + \int_0^T\int_\Omega a(x)\,|\nabla u(t,x)|\,\log\big(e + |\nabla(u(t,x))|\big)\,dx\,dt < \infty, \tag{5}$$

with $p > 1$ fixed and where the Lebesgue space $L^q_t L^p_a$ and more generally the Sobolev space $L^q_t W^{1,p}_a$ are defined by the norms

$$\|f\|_{L^q_t L^p_a} := \left\| \left(\int_\Omega |f|^p \, a(x)\, dx \right)^{1/p} \right\|_{L^q([0,\ T])} < \infty,$$

$$\|f\|_{L^q_t W^{1,p}_a} := \left\| \left(\int_\Omega (|f|^p + |\nabla f|^p) \, a(x)\, dx \right)^{1/p} \right\|_{L^q([0,\ T])} < \infty.$$

Because we do not have direct bounds on $\operatorname{div} u$ or even on ∇u as a may vanish, the standard theory of renormalized solutions cannot be applied to provide regularity (compactness of the solutions) or uniqueness. Concerning the boundary conditions on the velocity field, the anelastic constraint (2) and the integrability assumption on the velocity field allow to consider velocity fields satisfying the boundary condition in (2) in a weak sense, see for instance [25].

We propose here to extend the method introduced in [10] to this degenerate PDE system (1)–(3) through an appropriate three level weights control. This helps to encode quantitative stability estimates when approaching the degenerate constraint by a non-degenerate one: a standard procedure when you want to approximate a degenerate PDE. The conclusion will be existence of renormalized solution to the advective equations with degenerate anelastic constraint, as per

Theorem 1. *We have stability and existence of renormalized solutions:*
1. (Stability) For any C^1 sequences a_ε, α_ε, u_ε and a sequence of Lipschitz open domains Ω_ε with

- *a_ε is bounded from below, $\inf_{\Omega_\varepsilon} a_\varepsilon > 0$, and we have the divergence condition*

$$div\,(a_\varepsilon\, u_\varepsilon) = 0, \tag{6}$$

- *a_ε, α_ε, u_ε satisfy (2) and (4)-(5) uniformly in ε: $\sup_\varepsilon A(\alpha_\varepsilon, a_\varepsilon) + \sup_\varepsilon \|u_\varepsilon\|_{a_\varepsilon} < \infty$,*
- *Ω_ε converges to Ω for the Hausdorff distance on sets and $\|a_\varepsilon - a\|_{L^1(\Omega_\varepsilon \cap \Omega)} \to 0$ as $\varepsilon \to 0$,*

and for any sequence of initial data ϕ^0_ε uniformly bounded in $L^\infty(\mathbb{R}^d)$ and compact in $L^1(\mathbb{R}^d)$, consider the unique Lipschitz solution ϕ_ε to

$$a_\varepsilon\,(\partial_t \phi_\varepsilon + u_\varepsilon \cdot \nabla \phi_\varepsilon) = 0, \quad \text{in } \Omega_\varepsilon, \tag{7}$$

with boundary condition

$$a_\varepsilon\, u_\varepsilon \cdot n = 0, \quad \text{on } \partial\Omega_\varepsilon. \tag{8}$$

Then ϕ_ε is compact in $L^\infty_t L^2_{a_\varepsilon}$ and converges to a renormalized solution to (1) with (2).
2. (Existence) Let ϕ_0 be in $L^\infty(\Omega)$ and (a, α, u) satisfy (2) and the bounds (4) and (5). Then there exists a renormalized solution ϕ of (1) with initial data (3).

We present a possible strategy at the end of the article to use our techniques to prove that any weak solution is a renormalized solution and thus provide uniqueness of the solution; the full argument would however go beyond the limited scope of these proceedings.

The main ingredient to prove Theorem 1, is to obtain uniform regularity estimates on ε. This is done in two steps: First introducing appropriate weights in section 2 and then propagating regularity in the next section. We can then construct a sequence of solutions ϕ^ε for the approximate coefficients a_ε and obtain the renormalized solution as the strong limit.

We conclude the manuscript by showing the existence of global renormalized equations for the lake equations and presenting also a formal derivation of the model from compressible equation from Fluid Mechanics. Since our method is based on a doubling of variable argument, we make abundant use of notations like $u^x = u(t,x)$ to keep track of the physical variable (comparing u^x and u^y for $x \neq y$) whereas the value of the time variable is usually obvious.

2. Three-level weights procedure and properties

The estimates in this part hold for general coefficients with appropriate renormalized solutions but will later be used with the approximate coefficients a_ε, α_ε and the velocity u_ε.

As in [10], we introduce auxiliary equations that will help to identify the appropriate trajectories where the flow has some regularity. In this paper, we do it in three steps to control trajectories : where α is very small, where $|u|$ is large and where oscillations in the velocity field occur. More precisely, we define w_a solution to

$$\partial_t w_a + u\cdot\nabla w_a = -\gamma\,\frac{|u\cdot\nabla\alpha|}{\alpha}\,w_a, \qquad w_a|_{t=0} = (\alpha(x))^\gamma. \tag{9}$$

The weight w_a controls which trajectories can get close to points where α (and hence a) are very small. Next we introduce w_u solution to

$$\partial_t w_u + u\cdot\nabla w_u = -w_u\,|u(t,x)|\,\frac{1+\int_0^t|\nabla u(s,x)|\,ds}{1+\int_0^t|u(s,x)|\,ds}, \qquad w_u|_{t=0} = 1, \tag{10}$$

which controls trajectories going near points where $|u|$ is large. Finally we define our main weight, controlling oscillations in the velocity field

$$\partial_t w + u\cdot\nabla w = -D\ w, \qquad w|_{t=0} = 1, \tag{11}$$

with

$$D = \lambda\Big[\frac{M\,|\nabla(\alpha\,u)|}{\alpha} + (M\,|\nabla\alpha|(x))^\theta\,|u^x|^\theta + |\alpha^x|^{-\theta^*}\Big]$$

for some constants λ, θ and θ^* (chosen later on) respectively such that $\lambda > 0,\ 1/\theta^* = 1 - 1/\theta$ with $p > \theta > 1$.

Observe that for general a, α and u only satisfying (4)-(5), we are at this point incapable of ensuring that there exist renormalized solutions to Eqs (9), (10), (11); in fact this would only follow from a first application of our method.

However assuming that such solutions exist, we can easily investigate their properties, summarized in the following

Lemma 2. *Assume that* (4) *holds and that u satisfies* (2) *and* (5). *Then*

- *Consider w_a a renormalized solution to* (9). *One has that*

$$\begin{aligned}&0 \le w_a(t,x) \le (\alpha(x))^\gamma \le (a(x))^\gamma,\\ &\int_\Omega a(x)\,w_u(t,x)\,|\log w_a(t,x)|\,dx\\ &\qquad\qquad \le C\,\gamma\,\Big((1+\|u\|_{L^1_t L^p_a})\,\|\nabla\log\alpha\|_{L^r_a(\Omega)} + \|a\,\log\alpha\|_{L^1(\Omega)}\Big).\end{aligned} \tag{12}$$

- *Consider w_u a renormalized solution to* (10). *One has that*

$$0 \le w_u(t,x) \le \frac{1}{1+\int_0^t|u(s,x)|\,ds}, \qquad \int_\Omega a(x)\,|\log w_u(t,x)|\,dx \le C_T\,\|u\|_a. \tag{13}$$

- *Finally consider w a renormalized solution to* (11). *One has that*

$$
\begin{aligned}
&0 \le w(t,x) \le 1,\\
&\int_\Omega a(x)\, w_a(t,x)\, |\log w(t,x)|\, dx \le C\, T + C\, \|u\|_{L^p_a}^\theta\, \|\nabla \alpha^{1/p^*}\|_{L^q}^\theta \\
&\qquad + C \int_0^T \int_\Omega a\, |\nabla u|\, \log(e + |\nabla u|)\, dx\, dt.
\end{aligned}
\tag{14}
$$

Lemma 2 in particular shows that $w_u > 0$ a-almost everywhere, that $w_a > 0$ $a\, w_u$-almost everywhere and finally that $w > 0$ $a\, w_a$-almost everywhere; and by the previous points, $w_a > 0$ and $w > 0$ a-almost everywhere as well.

Proof. 1) **Estimates on w_u.**
1-1) *Pointwise control.* Since $w = 1$ identically at $t = 0$ and $D \ge 0$, one trivially has that $0 \le w \le 1$. The other estimates are less straightforward and we start by proving them on w_u. Define

$$\varphi(t,x) = -\log(1 + \int_0^t |u(s,x)|\, ds),$$

and notice that

$$\partial_t \varphi + u \cdot \nabla \varphi = -\frac{|u(t,x)|}{1 + \int_0^t |u(s,x)|\, ds} - \frac{u(t,x) \cdot \int_0^t \nabla_x u(s,x) \cdot \frac{u(s,x)}{|u(s,x)|}\, ds}{1 + \int_0^t |u(s,x)|\, ds},$$

while $\varphi(t = 0, x) = 0$. Therefore by (10), one has that

$$\partial_t \log w_u + u \cdot \nabla_x \log w_u \le \partial_t \varphi + u \cdot \nabla \varphi.$$

By the maximum principle since $\log w_u = \varphi$ at $t = 0$, we have that $\log w_u \le \varphi$ and by taking the exponential

$$w_u \le e^\varphi = \frac{1}{1 + \int_0^t |u(s,x)|\, ds}.$$

1-2) *A* log-*control on w_u.* Using again the equation (10), and since $\mathrm{div}(a\, u) = 0$, we have that

$$\frac{d}{dt} \int_\Omega a(x)\, |\log w_u(t,x)|\, dx = \int_\Omega a(x)\, |u(t,x)|\, \frac{1 + \int_0^t |\nabla u(s,x)|\, ds}{1 + \int_0^t |u(s,x)|\, ds}\, dx.$$

Therefore by the definition of φ

$$\int_\Omega a(x)\, |\log w_u(t_0,x)|\, dx = -\int_0^{t_0} \int_\Omega \partial_t \varphi(t,x)\, \left(a + \int_0^t a(x)\, |\nabla u(s,x)|\, ds \right)\, dx\, dt.$$

Integrating by part in time

$$
\begin{aligned}
\int_\Omega a(x)\, |\log w_u(t_0,x)|\, dx = &-\int_0^t \int_\Omega a\, \partial_t \varphi(t,x) + \int_0^{t_0} \int_\Omega \varphi(t,x)\, a(x)\, |\nabla u(t,x)|\, dx\, dt \\
&- \int_\Omega a(x)\, \varphi(t_0,x) \int_0^{t_0} |\nabla u(s,x)|\, ds\, dx.
\end{aligned}
$$

Remark that the first term reads

$$0 \le -\int_0^t \int_\Omega a\, \partial_t \varphi(t,x) \le \|u\|_{L^1_t L^1_a}.$$

Note that the second term in the right-hand side is negative. For the last term, we use the well-known convex inequality, $x\,y \le x\log(e+x) + e^y$ for $x,\,y \ge 0$ to bound

$$
\begin{aligned}
&-\int_\Omega a(x)\,\varphi(t_0,x)\int_0^{t_0}|\nabla u(s,x)|\,ds\,dx\\
&\quad\le \int_0^{t_0}\int_\Omega a(x)\,\left(|\nabla u(s,x)|\,\log(e+|\nabla u(s,x)|)+e^{|\varphi(t_0,x)|}\right)\,ds\,dx\\
&\quad\le \int_0^{t_0}\int_\Omega a(x)\,\left(|\nabla u(s,x)|\,\log(e+|\nabla u(s,x)|)+1+\int_0^{t_0}|u(r,x)|\,dr\right)\,ds\,dx,
\end{aligned}
$$

again by the definition of φ. Hence

$$
\begin{aligned}
&\int_\Omega a(x)\,|\log w_u(t_0,x)|\,dx\\
&\quad\le C_T\,\left(\|u\|_{L^1_tL^1_a}+\int_0^{t_0}\int_\Omega a(x)\,|\nabla u(s,x)|\,\log(e+|\nabla u(s,x)|)\,dx\,ds\right).
\end{aligned}
$$

2) **Estimates on w_a.**

2.1) *Pointwise control on w_a.* We now turn to the estimate on w_a. First note that

$$\partial_t\alpha + u\cdot\nabla\alpha = u\cdot\nabla\alpha \ge -\frac{|u\cdot\nabla\alpha|}{\alpha}\,\alpha,$$

and therefore, just as for w_u, by the maximum principle $\log w_a \le \gamma\log\alpha$ which leads to

$$w_a(t,x)\le(\alpha(x))^\gamma$$

and the other inequality as $\alpha\le a$.

1-2) *A* log*-control on w_a.* We also follow the same strategy to bound $|\log w_a|$ and obtain in a straightforward manner, using Eq. (10) on w_u, that

$$
\begin{aligned}
\int_\Omega a(x)\,w_u(t_0,x)\,|\log w_a(t_0,x)|\,dx \le&\gamma\int_0^{t_0}\int_\Omega a(x)\,w_u\,|u|\,|\nabla\log\alpha|\,dx\,dt\\
&+\int_\Omega a\,|\log w_a(t=0,x)|\,dx.
\end{aligned}
$$

From the initial data on w_a, $w_a(t=0,x)=(\alpha(x))^\gamma$, we have that

$$\int_\Omega a\,|\log w_a(t=0,x)|\,dx\le\gamma\int_\Omega a\,|\log\alpha|\,dx.$$

Furthermore since a and α do. not dependent on time, we also have that

$$
\begin{aligned}
\int_0^{t_0}\int_\Omega a(x)\,w_u\,|u|\,|\nabla\log\alpha|\,dx\,dt &\le \int_\Omega a(x)\,|\nabla\log\alpha|\int_0^{t_0}\frac{|u(t,x)|\,dt}{1+\int_0^t|u(s,x)|\,ds}\,dx\\
&=\int_\Omega a(x)\,|\nabla\log\alpha|\int_0^{t_0}\partial_t\log\left(1+\int_0^t|u(s,x)|\,ds\right)\,dt\,dx\\
&=\int_\Omega a(x)\,|\nabla\log\alpha|\,\log\left(1+\int_0^{t_0}|u(s,x)|\,ds\right)\,dx
\end{aligned}
$$

By bounding the log polynomially and a Hölder estimate, we deduce that

$$\int_\Omega a(x)\, w_u(t_0,x)\,|\log w_a(t_0,x)|\,dx\,dt \le \gamma\,\|\log\alpha\|_{L^1_a} + C_\mu\,\gamma\,\|u\|_{L^1_t L^p_a}\,\|\nabla\log\alpha\|_{L^{1+\mu}_a},$$

for any $\mu > 0$. Choosing μ s.t. $1+\mu \le r$ one has

$$\int_\Omega a(x)\, w_u(t_0,x)\,|\log w_a(t_0,x)|\,dx\,dt \le \gamma\,\|\log\alpha\|_{L^1_a} + \gamma\,\|u\|_{L^1_t L^p_a}\,\|\nabla\log\alpha\|_{L^r_a}. \tag{15}$$

2) **Estimates on w.** The point wise estimate on w is straightforward due to the damping term and the initial data. We now turn to the last estimate on $\log w$. Following similar calculations with Eqs. (11) and (9), we have that

$$\frac{d}{dt}\int_\Omega a(x)\, w_a(t,x)\,|\log w(t,x)|\,dx \le \int_\Omega a(x)\, w_a(t,x)\, D(t,x)\,dx.$$

Since $w_a(t,x) \le (\alpha(x))^\gamma$, if $\gamma \ge \theta^*$, one has from the definition of D in (11) that

$$\frac{d}{dt}\int_\Omega a(x)\, w_a(t,x)\,|\log w(t,x)|\,dx \le \int_\Omega a(x)\,\Big(M\,|\nabla(\alpha u)| + (M|\nabla\alpha|)^\theta |u|^\theta (\alpha)^\gamma + 1\Big)\,dx.$$

We may simply bound

$$\int_\Omega a(x)\,(M|\nabla\alpha|)^\theta\,|u|^\theta(\alpha)^\gamma\,dx \le C\,\|\nabla\alpha\|^\theta_{L^q}\|u\|^\theta_{L^p_a},$$

with $q > \theta$ and recalling the maximal function is bounded on L^q as $q > 1$. For the other term, by the standard properties of the maximal function, one has that

$$\begin{aligned}\int_0^T\int_\Omega a(x)\, M|\nabla(\alpha\, u)|(t,x)\,dx\,dt &\le C\int_0^T\int_\Omega M|\nabla(\alpha\, u)|(t,x)\,dx\,dt\\ &\le C\int_0^T \||\nabla(\alpha\, u)(t,.)|\,\|_{\mathcal{H}^1}\,dt,\end{aligned}$$

where $\mathcal{H}^1$ is the classical Hardy space. Since $|\nabla(\alpha u)|$ is always positive and Ω is bounded, this Hardy norm reduces to a $L\log L$ estimate

$$\|\,|\nabla(\alpha\, u)|\,\|_{\mathcal{H}^1} \sim C\,\left(\int_\Omega |\nabla(\alpha\, u)|\,\log(e+|\nabla(\alpha\, u)|)\,dx\right).$$

This is of course slightly non-optimal as we are losing possible cancellations in $\nabla(\alpha\, u)$, but necessary here if we want to keep positive weights. Of course since $\nabla(\alpha\, u) = u\nabla\alpha + \alpha\,\nabla u$, we have for example by the properties of the log and Hölder estimates that

$$\begin{aligned}\int_\Omega |\nabla(\alpha u)|\,\log(e+|\nabla(\alpha u)|))\,dx \le& C\int_\Omega \alpha\,|\nabla u|\,\log(e+|\nabla u|)\,dx\\ &+ C\,\|u\|^\theta_{L^p_\alpha}\,\|\nabla\alpha^{1/p^*}\|^\theta_{L^q},\end{aligned}$$

where one needs $q > p^*$. Therefore since $\alpha \le a$, we finally find that

$$\begin{aligned}\int_\Omega a(x)\, w_a(t,x)\,|\log w(t,x)|\,dx \le& C\,T + C\,\|u\|^\theta_{L^p_a}\,\|\nabla\alpha^{1/p^*}\|^\theta_{L^q}\\ &+ C\int_0^T\int_\Omega a\,|\nabla u|\,\log(e+|\nabla u|)\,dx\,dt.\end{aligned} \tag{16}$$

□

3. Compactness and quantitative regularity estimates

We consider here any renormalized solution to our main equation (1) and prove that it satisfies some quantified uniform regularity. As in the previous section those estimates will be applied for our approximate coefficients a_ε, α_ϵ as at this time we have not yet obtained renormalized solution in the general case.

3.1. **Regularity conditioned by the weights.** The first step is to propagate an adhoc semi-norms constructed with the weights, namely

Proposition 3. *Assume that ϕ is a renormalized solution to the transport equation in advective form* (1) *with constraints* (2)*. Let us define $\overline{a}$ corresponds to a on Ω and 0 on $\mathbb{R}^d\backslash\overline{\Omega}$. Assume as well that we have renormalized solutions w_a to* (9)*, w_u to* (10) *and w to* (11) *with λ large enough. One has that for any h and for $q > p^*$*

$$\int_{\mathbb{R}^{2d}} \overline{a}^x\,\overline{a}^y\,\frac{|\phi(t,x)-\phi(t,y)|}{(h+|x-y|)^d}\,w_a(t,x)\,w_u(t,x)\,w(t,x)\,w_a(t,x)\,w_u(t,y)\;w(t,y)\,dx\,dy$$
$$\le \int_{\mathbb{R}^{2\,d}} \overline{a}^x\,\overline{a}^y\,\frac{|\phi^0(x)-\phi^0(y)|}{(h+|x-y|)^d}\,dx\,dy$$
$$+\,C\,|\log h|^{1/2}\,\|\phi\|_{L^\infty}\,(\|u\|_a+\|u\|_a^\theta)\,(1+\|\nabla\alpha^{1/p^*}\|_{L^q})^\theta.$$

Proof. We skip the bar on a to simplify calculations. Since ϕ is a renormalized solution, one has the non-linear identity

$$a^x a^y\Big[\partial_t|\phi^x-\phi^y| + u^x\cdot\nabla_x|\phi^x-\phi^y| + u^y\cdot\nabla_y|\phi^x-\phi^y|\Big] = 0.$$

Hence

$$\partial_t(a^x a^y|\phi^x-\phi^y|\,w^x\,w_a^x\,w_u^x\,w^y\,w_a^y\,w_u^y) + a^x a^y u^x\cdot\nabla_x(|\phi^x-\phi^y|\,w^x\,w_a^x\,w_u^x\,w^y\,w_a^y\,w_u^y)$$
$$+\,a^x a^y u^y\cdot\nabla_y(|\phi^x-\phi^y|\,w^x\,w_a^x\,w_u^x\,w^y\,w_a^y\,w_u^y)$$
$$\le -a^x a^y\,(D^x+D^y)\,|\phi^x-\phi^y|\,w^x\,w_a^x\,w_u^x\,w^y\,w_a^y\,w_u^y.$$

Multiplying by $(h+|x-y|)^d$ and integrating by parts yields

$$\frac{d}{dt}\int_{\mathbb{R}^{2d}} a^x a^y\frac{|\phi^x-\phi^y|}{(h+|x-y|)^d}\,w^x\,w_a^x\,w_u^x\,w^y\,w_a^y\,w_u^y\,dx\,dy$$
$$\le d\int_{\mathbb{R}^{2\,d}} a^x a^y\frac{|\phi^x-\phi^y|}{(h+|x-y|)^{d+1}}\,w^x\,w_a^x\,w_u^x\,w^y\,w_a^y\,w_u^y\,(u(t,x)-u(t,y))\cdot\frac{x-y}{|x-y|}\,dx\,dy$$
$$-\int_{\Omega^{2\,d}} a^x a^y\frac{|\phi^x-\phi^y|}{(h+|x-y|)^d}\,w^x\,w_a^x\,w_u^x\,w^y\,w_a^y\,w_u^y\,(D^x+D^y)\,dx\,dy.$$

As usual the main issue is the commutator estimate. As ∇u is only controlled when integrated against a, this is a more delicate issue. Indeed in principle u^x-u^y involves the values of ∇u between x and y whereas we only have the values of a at x and y. It is the reason why we need to introduce α, which has some regularity, and proceed with the following decomposition

$$\begin{aligned}\left|(u(t,x)-u(t,y))\cdot\frac{x-y}{|x-y|}\right| &\le \frac{1}{\alpha^x\,\alpha^y}\,\alpha^x\,\alpha^y\,|u(t,x)-u(t,y)|\\ &\le (\alpha^x)^{-1}\,(\alpha^y)^{-1}\,|\alpha^x\,u^x-\alpha^y\,u^y|\,\frac{\alpha^x+\alpha^y}{2}\\ &\quad+(\alpha^x)^{-1}\,(\alpha^y)^{-1}\,|\alpha^x-\alpha^y|\,\frac{\alpha^x\,|u^x|+\alpha^y\,|u^y|}{2}.\end{aligned}$$

By symmetry in x and y this leads to

(17)
$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^{2\,d}} a^x a^y\frac{|\phi^x-\phi^y|}{(h+|x-y|)^d}\,w^x\,w_a^x\,w_u^x\,w^y\,w_a^y\,w_u^y\,dx\,dy\\ &\le d\int_{\mathbb{R}^{2\,d}} a^x a^y\frac{|\phi^x-\phi^y|}{(h+|x-y|)^{d+1}}\,w^x\,w_a^x\,w_u^x\,w^y\,w_a^y\,w_u^y\,|\alpha^x\,u^x-\alpha^y\,u^y|\,\frac{\alpha^x+\alpha^y}{\alpha^x\,\alpha^y}\,dx\,dy\\ &\quad+d\int_{\mathbb{R}^{2\,d}} a^x a^y\frac{|\phi^x-\phi^y|}{(h+|x-y|)^{d+1}}\,w^x\,w_a^x\,w_u^x\,w^y\,w_a^y\,w_u^y\,|\alpha^x-\alpha^y|\,\frac{\alpha^x\,|u^x|+\alpha^y\,|u^y|}{\alpha^x\,\alpha^y}\,dx\,dy\\ &\quad-\int_{\Omega^{2\,d}} a^x a^y\frac{|\phi^x-\phi^y|}{(h+|x-y|)^d}\,w^x\,w_a^x\,w_u^x\,w^y\,w_a^x\,w_u^y\,(D^x+D^y)\,dx\,dy.\end{aligned}$$

We now appeal to the technical lemmas that have already been used in [10] to control the difference u^x-u^y.

Lemma 4. *There exists $C>0$ s.t. for any $f\in W^{1,1}(\mathbb{R}^d)$, one has*

$$|f(x)-f(y)|\le C\,|x-y|\,(D_{|x-y|}f(x)+D_{|x-y|}f(y)),$$

where we denote

$$D_h f(x)=\frac{1}{h}\int_{|z|\le h}\frac{|\nabla f(x+z)|}{|z|^{d-1}}\,dz.$$

A full proof of such well known result can for instance be found in [13] in a more general setting namely $f\in BV$. Through a simple dyadic decomposition, one may also immediately deduce that

$$D_h\,f(x)\le C\,M|\nabla f|(x), \tag{18}$$

where M denotes the usual maximal operator, and thus recovering the classical bound

$$|f(x)-f(y)|\le C\,|x-y|\,(M|\nabla f|(x)+M|\nabla f|(y)). \tag{19}$$

Applying Lemma 4 to Eq. (17), we find, by symmetry in x and y that

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^{2\,d}} a^x a^y\frac{|\phi^x-\phi^y|}{(h+|x-y|)^d}\,w^x\,w_a^x\,w_u^x\,w^y\,w_a^y\,w_u^y\,dx\,dy\\ &\le C\int_{\mathbb{R}^{2\,d}} a^x a^y\frac{|\phi^x-\phi^y|}{(h+|x-y|)^d}\,w^x\,w_a^x\,w_u^x\,w^y\,w_a^y\,w_u^y\,(D_{|x-y|}(\alpha\,u)(x)+D_{|x-y|}(\alpha\,u)(y))\,\frac{dx\,dy}{\alpha^x}\\ &\quad+C\int_{\mathbb{R}^{2\,d}} a^x a^y\frac{|\phi^x-\phi^y|}{(h+|x-y|)^d}\,w^x\,w_a^x\,w_u^x\,w^y\,w_a^y\,w_u^y\,(D_{|x-y|}\alpha(x)+D_{|x-y|}\alpha(y))\,\frac{|u^x|}{\alpha^y}\,dx\,dy\\ &\quad-\int_{\Omega^{2\,d}} a^x a^y\frac{|\phi^x-\phi^y|}{(h+|x-y|)^d}\,w^x\,w_a^x\,w_u^x\,w^y\,w_a^y\,w_u^y\,(D^x+D^y)\,dx\,dy.\end{aligned}$$

We recall the definition of the penalization D^x

$$D^x = \lambda \left(\frac{M\,|\nabla(\alpha\,u)|(x)}{\alpha^x} + (M\,|\nabla\alpha|(x))^\theta\,|u^x|^\theta + |\alpha^x|^{-\theta^*} \right)$$

with $\lambda > 0$ chosen large enough. Since $v\,w \leq v^\theta + w^{\theta^*}$, one has that

$$D_{|x-y|}\alpha(x)\,\frac{|u^x|}{\alpha^y} \leq (M\,|\nabla\alpha|(x))^\theta\,|u^x|^\theta + |\alpha^y|^{-\theta^*}.$$

By the bound (18) with some symmetry in x and y, and using λ large enough, we therefore obtain that

(20)

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^{2d}} a^x a^y \frac{|\phi^x-\phi^y|}{(h+|x-y|)^d}\,w^x\,w_a^x\,w_u^x\,w^y\,w_a^y\,w_u^y\,dx\,dy\\
&\quad\leq C\int_{\mathbb{R}^{2\,d}} a^x a^y \frac{|\phi^x-\phi^y|}{(h+|x-y|)^d}\,w^x\,w_a^x\,w_u^x\,w^y\,w_a^y\,w_u^y\,(D_{|x-y|}(\alpha\,u)(y)-D_{|x-y|}(\alpha\,u)(x))\,\frac{dx\,dy}{\alpha^x}\\
&\qquad+C\int_{\mathbb{R}^{2\,d}} a^x a^y \frac{|\phi^x-\phi^y|}{(h+|x-y|)^d}\,w^x\,w_a^x\,w_u^x\,w^y\,w_a^y\,w_u^y\,(D_{|x-y|}\alpha(y)-D_{|x-y|}\alpha(x))\,\frac{|u^x|}{\alpha^y}\,dx\,dy.
\end{aligned}$$

Recalling now Lemma 2, we have that $w_a^x \leq \alpha^x$. Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}} a^x a^y \frac{|\phi^x-\phi^y|}{(h+|x-y|)^d}\,w^x\,w_a^x\,w_u^x\,w^y\,w_a^y\,w_u^y\,(D_{|x-y|}(\alpha\,u)(y)-D_{|x-y|}(\alpha\,u)(x))\,\frac{dx\,dy}{\alpha^x}\\
&\quad\leq C\|\phi\|_{L^\infty}\int_{\mathbb{R}^{2\,d}} \frac{|D_{|x-y|}(\alpha\,u)(y)-D_{|x-y|}(\alpha\,u)(x)|}{(h+|x-y|)^d}\,dx\,dy\\
&\quad\leq C\|\phi\|_{L^\infty}\int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}^d}\int_0^R |D_\rho(\alpha\,u)(x+\rho\,w)-D_\rho(\alpha\,u)(x)|\,\frac{d\rho}{h+\rho}\,d\rho\,dx\,dw,
\end{aligned}$$

by a direct change of variables to polar coordinates in $y-x$ and where R is the diameter of Ω. This leads to a square function type of estimates as by Cauchy-Schwartz

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}} a^x a^y \frac{|\phi^x-\phi^y|}{(h+|x-y|)^d}\,w^x\,w_a^x\,w_u^x\,w^y\,w_a^y\,w_u^y\,(D_{|x-y|}(\alpha\,u)(y)-D_{|x-y|}(\alpha\,u)(x))\,\frac{dx\,dy}{(\alpha^x)^\theta}\\
&\quad\leq C\,\|\phi\|_{L^\infty}\int_{\mathbb{S}^{d-1}}|\log h|^{1/2}\int_{\mathbb{R}^d}\left(\int_0^R |D_\rho(\alpha\,u)(x+\rho\,w)-D_\rho(\alpha\,u)(x)|^2\,\frac{d\rho}{h+\rho}\right)^{1/2}dx\,dw.
\end{aligned}$$

We now recall the classical estimate (see for example the remark on page 159 in [34])

Lemma 5. *For any $1 < p < \infty$, any family L_ρ of kernels satisfying for some $s > 0$*

$$\int L_\rho = 0, \quad \sup_\rho\,(\|L_\rho\|_{L^1} + \rho^s\,\|L_\rho\|_{W^{s,1}}) \leq C_L, \quad \sup_\rho\,\rho^{-s}\int |z|^s\,|L_\rho(z)|\,dz \leq C_L. \tag{21}$$

Then there exists $C > 0$ depending only on C_L above s.t. for any f in the Hardy space $\mathcal{H}^1(\Omega)$

$$\int_{\mathbb{R}^d}\left(\int_0^1 |L_\rho\star f(x)|^2\,\frac{d\rho}{h+\rho}\right)^{1/2}dx \leq C\,\|f\|_{\mathcal{H}^1}, \tag{22}$$

whereas if $f \in L^p$ with $1 < p < \infty$

$$\int_{\mathbb{R}^d}\left(\int_0^1 |L_\rho\star f(x)|^2\,\frac{d\rho}{h+\rho}\right)^{p/2}dx \leq C_p\,\|f\|_{L^p}^p. \tag{23}$$

Observe that obviously

$$D_\rho f = \bar L_\rho \star |\nabla f|, \quad \bar L_\rho(x) = \frac{1}{\rho\,|x|^{d-1}}\,\mathbb{I}_{|x|\leq\rho} = \rho^{-d}\,\bar L(x/\rho) \quad \text{with } \bar L(x) = \frac{1}{|x|^{d-1}}\,\mathbb{I}_{|x|\leq 1}.$$

Hence defining $L_\rho(x) = \bar L_\rho(x) - \bar L_\rho(x+\rho\,w)$, we can easily check that L_ρ satisfies the assumptions of Lemma 5. This proves that

$$\int_{\mathbb{R}^{2d}} a^x a^y \frac{|\phi^x-\phi^y|}{(h+|x-y|)^{d+1}}\, w^x\, w_a^x\, w^y\, w_a^y\, (D_{|x-y|}(\alpha\, u)(y) - D_{|x-y|}(\alpha\, u)(x))\,\frac{dx\,dy}{\alpha^x}$$
$$\leq C\,\|\phi\|_{L^\infty}\,|\log h|^{1/2}\,\|\,|\nabla(\alpha\, u)|\,\|_{\mathcal{H}^1(\Omega)}.$$

We now follow the exact same steps as for the bound at the end of the proof of Lemma 2. Note that here it would be easier to use the cancellations in $\nabla(\alpha u)$ by being more precise in Lemma 4 and using an exact representation instead of a bound. For simplicity though, here we have kept the more direct version of Lemma 4. Hence we have that

$$\int_\Omega |\nabla(\alpha u)|\,\log(e+|\nabla(\alpha u)|)\,dx \leq C\int_\Omega a\,|\nabla u|\,\log(e+|\nabla u|)\,dx$$
$$+\,C\,\|u\|_{L^p_a}^\theta\,\|\nabla\alpha^{1/p^*}\|_{L^q}^\theta,$$

where again one needs $q>p^*$. This lets us conclude that

$$\int_0^T\int_{\mathbf{R}^{2d}} a^x a^y \frac{|\phi^x-\phi^y|}{(h+|x-y|)^{d}}\, w^x\, w_a^x\, w^y\, w_a^y\, (D_{|x-y|}(\alpha\, u)(y) - D_{|x-y|}(\alpha\, u)(x))\,\frac{dx\,dy\,dt}{\alpha^x}$$
$$\leq C_T\,\|\phi\|_{L^\infty}\,|\log h|^{1/2}\,[\|u\|_a + \|u\|_{L^p_a}^\theta\|\nabla\alpha^{1/p^*}\|_{L^q}^\theta]. \tag{24}$$

We apply the same strategy to the other term in the bound (20). We again start using that $w_a^y\leq\alpha^y$ to obtain that

$$\int_0^T\int_{\mathbf{R}^{2d}} a^x a^y \frac{|\phi^x-\phi^y|}{(h+|x-y|)^{d}}\, w^x\, w_a^x\, w_u^x\, w^y\, w_a^y\, w_u^y\, (D_{|x-y|}\alpha(y) - D_{|x-y|}\alpha(x))\,\frac{|u^x|\,dx\,dy\,dt}{\alpha^y}$$
$$\leq \|\phi\|_{L^\infty}\int_{\mathbf{R}^{2d}}\frac{|D_{|x-y|}\alpha(y)-D_{|x-y|}\alpha(x)|}{(h+|x-y|)^d}\int_0^T w_u^x\,a^x\,|u^x|\,dt\,dx\,dy,$$

since α is independent of time. By Prop. 2, $w_u\leq 1/(1+\int_0^t|u(s,x)|\,ds$ and hence

$$\int_0^T w_u(t,x)\,|u(t,x)|\,dt \leq \int_0^T\frac{|u(t,x)|}{1+\int_0^t|u(s,x)|\,ds}\,dt = \int_0^T\partial_t\log\left(1+\int_0^t|u(s,x)|\,ds\right)\,dt$$
$$=\log\left(1+\int_0^T|u(s,x)|\,ds\right).$$

Choose now any $\mu>0$ and bound

$$\log\left(1+\int_0^T|u(s,x)|\,ds\right)\leq C_\mu\,\left(1+\int_0^T|u(s,x)|\,ds\right)^{\mu/(1+\mu)},$$

so that by Hölder since $1-1/(1+\mu)=\mu/(1+\mu)$

$$\int_0^T \int_{\mathbf{R}^{2d}} a^x a^y \frac{|\phi^x-\phi^y|}{(h+|x-y|)^d}\, w^x\, w_a^x\, w_u^x\, w^y\, w_a^y\, w_u^y\, (D_{|x-y|}\alpha(y)-D_{|x-y|}\alpha(x))\, \frac{|u^x|\,dx\,dy\,dt}{\alpha^y}$$
$$\leq C_\mu\, \|\phi\|_{L^\infty}\, \|u\|_{L^1_t L^1_a}\, |\log h|^{\mu/(1+\mu)} \left(\int_{\mathbf{R}^{2d}} \frac{|D_{|x-y|}\alpha(y)-D_{|x-y|}\alpha(x)|^{1+\mu}}{(h+|x-y|)^d}\, dx\,dy\right)^{1/(1+\mu)}.$$

We can now apply Lemma 5 for $f\in L^p$, and find similarly that

$$\int_{\mathbf{R}^{2d}} \frac{|D_{|x-y|}\alpha(y)-D_{|x-y|}\alpha(x)|^{1+\mu}}{(h+|x-y|)^d}\, dx\,dy \leq C_\mu\, |\log h|^{(1-\mu)/2}\, \|\nabla\alpha\|_{L^{1+\mu}}^{1+\mu}.$$

This leads to
(25)

$$\int_0^T \int_{\mathbf{R}^{2d}} a^x a^y \frac{|\phi^x-\phi^y|}{(h+|x-y|)^{d+1}}\, w^x\, w_a^x\, w_u^x\, w^y\, w_a^y\, w_u^y\, (D_{|x-y|}\alpha(y)-D_{|x-y|}\alpha(x))\, \frac{|u^x|\,dx\,dy\,dt}{\alpha^y}$$
$$\leq C_\mu\, \|\phi\|_{L^\infty}\, \|u\|_{L^1_t L^1_a}\, |\log h|^{1/2}\, \|\nabla\alpha\|_{L^{1+\mu}}.$$

Choosing μ small with $1+\mu\leq q$ and combining (25) with (24) in (20), we finally conclude that

$$\frac{d}{dt}\int_{\mathbf{R}^{2d}} a^x a^y \frac{|\phi^x-\phi^y|}{(h+|x-y|)^d}\, w^x\, w_a^x\, w_u^x\, w^y\, w_a^y\, w_u^y\, dx\,dy$$
$$\leq C\, |\log h|^{1/2}\, \|\phi\|_{L^\infty}\, (\|u\|_a+\|u\|_a^\theta)\, (1+\|\nabla\alpha^{1/p^*}\|_{L^q})^\theta,$$

thus proving the proposition. □

3.2. **Our explicit regularity estimate.** By using a straightforward interpolation argument thanks to the previous controls obtained on the different weights w_u, w_a, w, we can now state our main result

Theorem 6. *Assume that* $(a,\,\alpha)$ *satisfy* (4) *and that* (2) *and* (5) *hold for* u*. Assume as well that we have renormalized solutions* w_a *to* (9)*,* w_u *to* (10) *and* w *to* (11)*. Consider now any renormalized solution to* (1) *and denote*

$$\|\phi^0\|_h=\frac{1}{|\log h|}\int_{\Omega^{2\,d}} a^x a^y \frac{|\phi^0(x)-\phi^0(y)|}{(h+|x-y|)^d}\, dx\,dy.$$

Then

$$\|\phi\|_h=\frac{1}{|\log h|}\int_{\Omega^{2d}} a^x a^y \frac{|\phi^x-\phi^y|}{(h+|x-y|)^d}\, dx\,dy \leq \frac{C}{|\log(\|\phi^0\|_h+|\log h|^{-1/2})|^{1/2}},$$

for some constant $C>0$ *depending only on the bounds on* $\|\alpha|_{L^\infty(\Omega)}$, $\|u\|_a$ *and* $\|\phi\|_{L^\infty((0,T)\times\Omega)}$.

Proof. The proof relies on a appropriate decomposition of the domain playing with sets constructed using intersection of the set $\{x,\,y\mid w_u(t,x)>\eta,\ w_u(t,y)>\eta\}$ or its complementary set with the set $\{x,\,y\mid w_a(t,x)>\eta',\ w_a(t,y)>\eta'\}$ and its complementary set and with the set $\{x,\,y\mid w(t,x)>\eta'',\ w(t,y)>\eta''\}$ and its complementary. More precisely, we write

$$\|\phi\|_h=\int_{\Omega^{2d}} \frac{|\phi^x-\phi^y|}{(h+|x-y|)^d}\, a^x\, a^y\, dx\,dy=\sum_{i=1}^4 \int_{I_i} \frac{|\phi^x-\phi^y|}{(h+|x-y|)^d}\, a^x\, a^y\, dx\,dy=\sum_{i=1}^4 J_j$$

with

$$I_1 = \{x,\, y \mid w_u(t,x) < \eta \text{ or } w_u(t,y) < \eta\},$$

$$I_2 = \{x,\, y \mid w_u(t,x) > \eta \text{ and } w_u(t,y) > \eta\} \cap \{x,\, y \mid w_a(t,x) < \eta' \text{ or } w_a(t,y) < \eta'\}$$

and denoting

$$I = \{x,\, y \mid w_u(t,x) > \eta \text{ and } w_u(t,y) > \eta\} \cap \{x,\, y \mid w_a(t,x) > \eta' \text{ and } w_a(t,y) > \eta'\},$$

with

$$I_3 = I \cap \{x,\, y \mid w(t,x) < \eta'' \text{ or } w(t,y) < \eta''\},$$

and

$$I_4 = I \cap \{x,\, y \mid w(t,x) > \eta'' \text{ and } w(t,y) > \eta''\}.$$

Note that it is straightforward that

$$0 \leq J_4 \leq \frac{1}{\eta^2 \eta'^2 \eta''^2} \int_{\Omega^{2d}} a^x\, a^y\, \frac{|\phi^x - \phi^y|}{(h + |x-y|)^d} w_a^x w_a^y w_u^x w_u^y w^x w^y \, dx\, dy.$$

Remark now that by symmetry, J_1 is bounded by

$$0 \leq J_1 \leq |\log h| \int_{x,\; w_u(t,x) \leq \eta} a^x (K_h \star a|\phi(t,x)| + K_h \star |a\phi|)\, dx,$$

where $K_h(x) = (h + |x|)^{-d}/|\log h|$ so $\|K_h\|_{L^1} = 1$. By Hölder estimate

$$\begin{aligned} &\int_{x,\; w_u(t,x) \leq \eta} a^x (K_k \star a|\phi(t,x)| + K_h \star |a\phi|)\, dx \\ &\qquad \leq C \|\phi\|_{L^\infty} \int_{x,\; w_u(t,x) \leq \eta} a\, dx. \end{aligned}$$

Now it suffices to note that

$$\int_{x,\; w_u(t,x) \leq \eta} a\, dx \leq \frac{1}{|\log \eta|} \int_{x,\; w_u(t,x) \leq \eta} |\log w_u(t,x)|\, a(t,x)\, dx \leq \frac{C}{|\log \eta|} \|u\|_a$$

to get an appropriate control. Similarly we get using properties of w_u and w_a

$$J_2 \leq \frac{C|\log h|}{\eta |\log \eta'|} \|\phi\|_{L^\infty} \int_\Omega a^x w_u^x |\log w_a^x|$$

We end the proof with the same kind of estimate on J_3 using properties of w_a and w, namely

$$J_3 \leq \frac{C|\log h|}{\eta' |\log \eta''|} \|\phi\|_{L^\infty} \int_\Omega a^x w_a^x |\log w^x|$$

Now using the bounds on $aw_u|\log w_a|$ and $aw_a|\log w|$ and the uniform bounds on u and α, and using Proposition 3 we get

$$\begin{aligned} &\sup_{t \in [0,T]} \frac{1}{|\log h|} \int_{\Omega^{2d}} a^x a^y \frac{|\phi^x - \phi^y|}{(h + |x-y|)^d}\, dx\, dy \\ &\quad \leq \frac{C}{\eta^2 \eta'^2 \eta''^2} \Big[\|\phi^0\|_h + |\log h|^{-1/2} \Big] + C \Big[\frac{1}{|\log \eta|} + \frac{1}{\eta |\log \eta'|} + \frac{1}{\eta' |\log \eta''|} \Big]. \end{aligned}$$

Optimizing in η, η', η'' (by choosing η in function of η' and η' in function of η'' and finally η'' in function of α and $|\log h|^{-1/2}$) we get the conclusion.

□

4. Stability and existence of renormalized solutions: Proof of Theorem 1.

4.1. **Stability of renormalized solutions.** Assume that we have been given sequences a_ε, α_ε and u_ε on a set Ω_ε which satisfy the assumptions specified in Theorem 1.

Since all terms are smooth, Eq. (7) has a unique Lipschitz solution ϕ_ε for any given initial data $\phi_\varepsilon^0 \in L^\infty(\Omega_\varepsilon)$. This solution is then obviously automatically renormalized. For the same reason we also trivially have solutions w_a to Eq. (9) with α_ε and u_ε and similarly for Eqs. (10) and (11). Of course while our solutions are smooth for a fixed ε, the main point is to derive and use uniform in ε bounds to obtain appropriate limits.

First define $\bar a_\varepsilon = a_\varepsilon$ on Ω_ε and extended by 0 on the whole of $\mathbb{R}^d$. Proceed similarly to define at the limit $\bar a$. Since a_ε is uniformly in L^∞, we can replace the convergence $\|a_\varepsilon - a\|_{L^1(\Omega_\varepsilon\cap\Omega)} \to 0$ and $\Omega_\varepsilon \to \Omega$ in Hausdorff distance by the simple

$$\|\bar a_\varepsilon - \bar a\|_{L^1(\mathbb{R}^d)} \longrightarrow 0.$$

From the uniform $L^\infty_t L^p_{a_\varepsilon}$ estimate for u_ε provided by (5) and $\sup_\varepsilon \|u_\varepsilon\|_{a_\varepsilon} < \infty$, we can extract a weak limit of $\bar a_\varepsilon^{1/p}\, u_\varepsilon$ in the whole space $\mathbb{R}^d$ and from the strong convergence of $\bar a_\varepsilon$, identify the limit as $\bar a\, u$ for some $u \in L^\infty_t L^p_a$:

$$\bar a_\varepsilon^{1/p}\, u_\varepsilon \longrightarrow \bar a^{1/p}\, u \quad \text{in } w-*\ L^\infty_t L^p(\mathbb{R}^d), \qquad u \in L^\infty_t L^p_a,$$

while for simplicity we still denote the extracted subsequence with ε. Since $\sup_\varepsilon \|\phi_\varepsilon^0\|_{L^\infty(\mathbb{R}^d)} < \infty$ then through renormalization $\sup_\varepsilon \|\phi_\varepsilon\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^d)} < \infty$, we may also extract a converging subsequence

$$\phi_\varepsilon \longrightarrow \phi \quad \text{in } w-*\ L^\infty(\mathbb{R}_+\times\mathbb{R}^d).$$

For any $\chi \in W^{1,\infty}(\mathbb{R})$ with $\chi(0) = 0$, $\chi(\phi_\varepsilon)$ still solves (7) by the chain rule for smooth functions. Choosing any test function $\psi \in C^\infty_c(\mathbb{R}^d)$, we deduce from (7) with the divergence condition (6) and the boundary conditions (8) the weak formulation

$$\frac{d}{dt}\int_{\Omega_\varepsilon} \chi(\phi_\varepsilon)\, a_\varepsilon\, \psi(x)\, dx - \int_{\Omega_\varepsilon} \chi(\phi_\varepsilon)\, a_\varepsilon\, u_\varepsilon\cdot\nabla_x\psi\, dx = 0.$$

The previous definition of $\bar a_\varepsilon$ and $\bar\phi_\varepsilon$ actually implies that this weak formulation is equivalent to the formulation in the whole space

$$\frac{d}{dt}\int_{\mathbb{R}^d} \chi(\phi_\varepsilon)\, \bar a_\varepsilon\, \psi(x)\, dx - \int_{\mathbb{R}^d} \chi(\phi_\varepsilon)\, \bar a_\varepsilon\, u_\varepsilon\cdot\nabla_x\psi\, dx = 0, \tag{26}$$

which is much simpler to use since the domain is now fixed. In that sense (26) implies the boundary condition (8) on $\partial\Omega_\varepsilon$ if one imposes that $\bar a_\varepsilon = 0$ out of Ω_ε. It is straightforward to check that $\bar a = 0$ out of Ω at the limit. Thus to prove that ϕ is a renormalized solution to (1) with (2) on the limiting set Ω, it is now enough to pass to the limit in (26).

Let us now first prove compactness in space on $\chi(\phi_\varepsilon)$ for any smooth function χ. This is exactly where our approach proves its use: We have all required assumptions to apply Theorem 6 and deduce from the compactness of ϕ_ε^0 and a_ε that

$$\limsup_{h\to 0} \frac{1}{|\log h|} \sup_\varepsilon \sup_t \int_{\mathbb{R}^{2d}} \bar a_\varepsilon^x\, \bar a_\varepsilon^y \frac{|\phi_\varepsilon^x - \phi_\varepsilon^y|}{(|x-y|+h)^d}\, dxdy \longrightarrow 0. \tag{27}$$

Note now that

$$a_\varepsilon^x a_\varepsilon^y(\phi_\varepsilon^x - \phi_\varepsilon^y) = (a_\varepsilon^x\phi_\varepsilon^x - a_\varepsilon^y\phi_\varepsilon^y)(a_\varepsilon^y + a_\varepsilon^x)/2 + (a_\varepsilon^y - a_\varepsilon^x)(a_\varepsilon^y\phi_\varepsilon^y + a_\varepsilon^x\phi_\varepsilon^x)/2$$

and that $|(a_\varepsilon^x\phi_\varepsilon^x - a_\varepsilon^y\phi_\varepsilon^y)| \le C(a_\varepsilon^x + a_\varepsilon^y)$ then get

$$|a_\varepsilon^x\phi_\varepsilon^x - a_\varepsilon^y\phi_\varepsilon^y|^2 \le C(a_\varepsilon^x a_\varepsilon^y|\phi_\varepsilon^x - \phi_\varepsilon^y| + |a_\varepsilon^y - a_\varepsilon^x|).$$

and therefore using (27) and compactness on a_ε, by the Rellich criterion this implies locally in space compactness of $a_\varepsilon\phi_\varepsilon$. Using the same procedure, it is possible to prove space compactness of $a_\varepsilon\chi(\phi_\varepsilon)$. We get compactness (in space and time) on $a_\varepsilon\chi(\phi_\varepsilon)$ using the renormalized equation which provides a control on $\partial_t(a_\varepsilon\chi(\phi_\varepsilon))$ allowing to use Aubin-Lions Lemma. Thus, up to a subsequence, we deduce that $a_\varepsilon\chi(\phi_\varepsilon)$ converges almost everywhere and thus $a_\varepsilon^{1-1/p}\chi(\phi_\varepsilon)$ converges almost everywhere using the compactness on a_ε. As ϕ_ε is uniformly bounded and therefore $\chi(\phi_\varepsilon)$ also, we get compactness of $a_\varepsilon^{1-1/p}\chi(\phi_\varepsilon)$. To conclude we just have to write $\chi(\phi_\varepsilon)a_\varepsilon u_\varepsilon = a_\varepsilon^{1-1/p}\chi(\phi_\varepsilon)\, a_\varepsilon^{1/p}u_\varepsilon$ and use the weak-star convergence of $a_\varepsilon^{1/p}u_\varepsilon$ in $L_t^\infty L_x^p$ and the strong convergence of $a_\varepsilon^{1-1/p}\chi(\phi_\varepsilon)$ in $L_t^1 L_x^q$ where $1/q + 1/p = 1$.

4.2. **Existence of renormalized solutions.** To obtain existence of renormalized solutions through a stability argument, it only remains to be able construct a sequence of approximations on which we may apply the previous stability argument.

In our case, given a, α and u which satisfy (2), (4) and (5), the first question is whether we can construct smooth a_ε, α_ε and u_ε which still satisfy the previous estimates uniformly in ε and where a_ε is bounded from below on Ω.

First define $\tilde\Omega_\varepsilon = \{\alpha > \varepsilon\}$. On $\tilde\Omega_\varepsilon$, one has that $a \ge \alpha > \varepsilon$; hence by (4), α belongs to a Sobolev space on a neighborhood of $\tilde\Omega_\varepsilon$ so that the boundary of $\tilde\Omega_\varepsilon$ is Lipschitz.

Define $\tilde a_\varepsilon = a$ on $\tilde\Omega_\varepsilon$ and $a = \varepsilon$ on $\Omega\setminus\tilde\Omega_\varepsilon$. Hence $\tilde a_\varepsilon$ may be discontinuous. Define similarly $\tilde\alpha_\varepsilon = \alpha$ on $\tilde\Omega_\varepsilon$ and ε outside. By the definition of $\tilde\Omega_\varepsilon$, $\tilde\alpha_\varepsilon$ does not jump on $\partial\tilde\Omega_\varepsilon$.

Note that $\tilde\alpha_\varepsilon$ satisfies (4) uniformly in ε, *i.e.* $\sup_\varepsilon A(\tilde\alpha_\varepsilon, \tilde a_\varepsilon) < \infty$; as for example

$$\tilde a_\varepsilon\,|\nabla\log\tilde\alpha_\varepsilon|^r = a\,|\nabla\log\alpha|^r\,\mathbb{I}_{\tilde\Omega_\varepsilon}.$$

Choose now a smooth and non-negative function χ s.t. $\chi(\xi/\varepsilon)$ is a good approximation of the Heaviside function with in particular $\chi(\xi/\varepsilon) = 0$ if $\xi \le \varepsilon$ and $\chi(\xi/\varepsilon) = 1$ if $\xi \ge 2\,\varepsilon$.

Define then $u_{\varepsilon,L} = \frac{u}{1+|u|/L}\,\chi(\alpha/\varepsilon)$. And observe that

$$\begin{aligned}\nabla u_{\varepsilon,L} &= \frac{\nabla u}{1+|u|/L}\,\chi(\alpha/\varepsilon) - \frac{u}{L\,(1+|u|/L)^2}\otimes\nabla u\cdot\frac{u}{|u|} + \frac{u}{1+|u|/L}\otimes\nabla\log\alpha\,\frac{\alpha}{\varepsilon}\,\chi'(\alpha/\varepsilon)\\ &= \frac{\nabla u}{1+|u|/L}\,\chi(\alpha/\varepsilon) + U_{\varepsilon,L} + D_{\varepsilon,L}.\end{aligned}$$

Since $\frac{\alpha}{\varepsilon}\,\chi'(\alpha/\varepsilon)$ is bounded uniformly and $\nabla\log\alpha \in L_a^r$, one has that $\|D_{\varepsilon,L}\|_{L_a^r} \le C\,L$ for some given constant C independent of ε and L. But note that $\nabla\log\alpha$ is independent of ε and L and hence equi-integrable in L_a^r while $\frac{\alpha}{\varepsilon}\,\chi'(\alpha/\varepsilon)$ converges to 0 in L^1 as $\varepsilon\to 0$. Hence for a fixed L, $D_{\varepsilon,L}\to 0$ as $\varepsilon\to 0$ for L fixed.

Therefore we can connect L and ε and choose L_ε s.t. $\|D_{\varepsilon,L}\|_{L_a^r}\to 0$ as $\varepsilon\to 0$. By the same type of equi-integrability arguments, we can show that

$$\int_\Omega a\,|U_{\varepsilon,L_\varepsilon}|\,\log(1+|U_{\varepsilon,L_\varepsilon}|)\,dx \longrightarrow 0, \quad \text{as } \varepsilon\to 0.$$

As a consequence $u_{\varepsilon,L_\varepsilon}$ still satisfies $\sup_\varepsilon\|u_{\varepsilon,L_\varepsilon}\|_a < \infty$. Hence since $u_{\varepsilon,L_\varepsilon}$ vanishes outside of $\tilde\Omega_\varepsilon$, it satisfies $\sup_\varepsilon\|u_{\varepsilon,L_\varepsilon}\|_{\tilde a_\varepsilon} < \infty$. We still need to correct the divergence and for this we solve the following elliptic equation

$$\mathrm{div}(a\,\nabla V_\varepsilon) = -a\,\mathrm{Tr}(U_{\varepsilon,L_\varepsilon} + D_{\varepsilon,L_\varepsilon}) \quad \text{in } \tilde\Omega_\varepsilon, \qquad V_\varepsilon = 0 \text{ on } \partial\tilde\Omega_\varepsilon.$$

Since a is bounded from below in $\tilde\Omega_\varepsilon$ and $\partial\Omega_\varepsilon$ is Lipschitz, this equation is well posed and we can extend V_ε to all Ω by taking $V_\varepsilon = 0$ in $\Omega \setminus \tilde\Omega_\varepsilon$. Furthermore by the previous bounds on $D_{\varepsilon,L_\varepsilon}$ and $L_{\varepsilon,L_\varepsilon}$, we have that V_ε converges to 0 in $W^{1,q}_{\tilde a_\varepsilon}$ for some $q > 1$. In dimension 2 for instance, the energy inequality would directly give this in $H^1_{\tilde a_\varepsilon}$.

We finally define $\tilde u_\varepsilon = u_{\varepsilon,L_\varepsilon} + \nabla V_\varepsilon$. From the construction, Eq. (6) holds for $\tilde a_\varepsilon$ and $\tilde u_\varepsilon$. The bounds in (4) and (5) are also satisfied uniformly in ε: $\sup_\varepsilon A(\tilde\alpha_\varepsilon, \tilde a_\varepsilon) + \sup_\varepsilon \|\tilde u_\varepsilon\|_{\tilde a_\varepsilon} < \infty$. Finally $\tilde a_\varepsilon$, $\tilde\alpha_\varepsilon$ and $\tilde u_\varepsilon$ all converge strongly.

Of course those coefficients are not yet smooth but this last step is the easiest and we only sketch it. By standard Sobolev approximation since $\tilde a_\varepsilon$ is now bounded from below, one may find α_ε and $\bar u_\varepsilon$ in $C^\infty(\Omega)$ but close to $\tilde\alpha_\varepsilon$ and $\tilde u_\varepsilon$ in the corresponding Sobolev spaces so that (4) and (5) still hold with weight $\tilde a_\varepsilon$ uniformly in ε.

One then approximates $\tilde a_\varepsilon$ by $a_\varepsilon \in C^\infty(\Omega)$, uniformly bounded and with $a_\varepsilon \geq \varepsilon$. In addition it is possible to choose $\|\tilde a_\varepsilon - a_\varepsilon\|_{L^1}$ small enough to obtain the uniform bounds (4) and (5): $\sup_\varepsilon A(\alpha_\varepsilon, a_\varepsilon) + \sup_\varepsilon \|u_\varepsilon\|_{a_\varepsilon} < \infty$.

We finally correct $u_\varepsilon = \bar u_\varepsilon + \nabla \bar V_\varepsilon$ as before to satisfy the divergence condition (6).

Once those approximated coefficients are constructed, we may directly apply our stability estimates to obtain at the limit a renormalized solution ϕ.

4.3. **Toward the uniqueness of weak solutions to** (1)**.** We conclude this section by briefly sketching a possible strategy to obtain the uniqueness of Eq. (1) by proving, as in the classical argument, that all weak solutions are also renormalized.

Since [18] this is usually performed by convolving Eq. (1) for any weak solution ϕ by some smooth kernel ρ_ε and showing that $\rho_\varepsilon(a\,\phi)$ still solves (1) with a right-hand side that is vanishing in L^1. This commutator estimate would require here that

$$\int_\Omega (u(t,x) - u(t,y))\cdot\nabla\rho_\varepsilon(x-y)\,a(y)\,\phi(t,y)\,dy \longrightarrow 0 \quad \text{in } L^1_t L^1_x \text{ as } \varepsilon \to 0. \tag{28}$$

One can then typically conclude by using Sobolev bounds on a. But since there is no $a(y)$ factor in the above integral and we only control ∇u in L^1_a, this cannot work here.

A second issue arises since (28) also usually requires a control on $\mathrm{div}\,u$ which is again unavailable.

Instead we would propose the following approach:

- Through a stability argument, obtain the existence of renormalized solutions to (9) and (10).
- Show that the commutator estimate (28) for $\phi = w_a\,w_u$ holds by using in particular that $w_a \leq a^\gamma$. The exact calculations here should be reminiscent of what were in essence other commutator estimates in the proof of Prop. 3.
- For any weak solution ϕ, use the previous point to prove that $\phi\,w_a\,w_u$ is also a solution to (1) with the corresponding added right-hand side from (9) and (10).
- Prove a commutator estimate like (28) but where ϕ is replaced by $\phi\,w_a\,w_u$.

There are obvious technical difficulties at each step and for this reason implementing such a strategy is beyond the limited scope of these proceedings.

5. An anelastic compressible equation coming from fluid mechanics

Let us present in this subsection a PDEs system occurring in fluid mechanics where the advective equation appears with a possible degenerate anelastic constraint. Then we will look more carefully on the two-dimensional in space lake equations where an advective equation with transport velocity satisfying the anelastic constraint appears.

i) *The anelastic constraint from compressible isentropic Euler equations.* This anelastic constraint appears when the Mach (or Froude) number tends to zero starting the compressible isentropic Euler equations with some heterogeneity F (bathymetry, stratification for instance). More precisely consider the following system

$$\partial_t \rho_\varepsilon + \mathrm{div}(\rho_\varepsilon u_\varepsilon) = 0$$

with

$$\partial_t(\rho_\varepsilon u_\varepsilon) + \mathrm{div}(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) + \frac{\nabla p(\rho_\varepsilon)}{\varepsilon^2} = \frac{\rho_\varepsilon \nabla F}{\varepsilon^2}$$

and the pressure law $p(\rho) = c\rho^\gamma$ (with two constants $c > 0$ and $\gamma > 1$) and where F is given and depends on the space variable (it represents heterogeneities in the environment). By letting formally ε to zero we get the following limit anelastic system

$$a\,(\partial_t u + u\cdot\nabla u) + a\nabla\pi = 0, \qquad \mathrm{div}(au) = 0 \qquad \text{where } a = \Big(\frac{\gamma-1}{c\,\gamma}\Big)^{1/(\gamma-1)} (F)^{1/(\gamma-1)}.$$

Therefore the anelastic constraint $\mathrm{div}(au) = 0$ actually accounts for the heterogeneity.

ii) *The lake equations.* This application concerns the so-called lake equation (under the rigid lid assumption) with possible vanishing topography. The PDEs is valid on a two-dimensional bounded domain Ω (the surface of the lake). This system reads

$$a(\partial_t u + u\cdot\nabla u + \nabla p) = 0 \qquad \text{with} \qquad \mathrm{div}(au) = 0 \text{ in } (0,T)\times\Omega$$

with respectively the boundary condition and the initial data

$$au\cdot n|_{(0,T)\times\partial\Omega} = 0, \qquad au|_{t=0} = m_0 \text{ in } \Omega$$

where a denotes the bathymetry and $u = (u_1, u_2)$ is a two-dimensional vector field which corresponds to the vertically averaging of the horizontal components of the velocity field $U = (U_1, U_2, W)$ in a three dimensional basin. Note that such system has been studied by [27] in the non-degenerate case and by [8], [25] and [31] in the degenerate case.

By reducing Ω to the support of a, we may assume that the bathymetry a is strictly positive in the domain Ω and possibly vanish on the shore $\partial\Omega$. Introducing the relative vorticity $\omega_R = \mathrm{curl}\, u/a$ where $\mathrm{curl}\, u = \partial_1 u_2 - \partial_2 u_1$, we check starting from the lake equation and dividing by a inside the domain that

$$\partial_t\omega_R + u\cdot\nabla\omega_R = 0 \text{ in } (0,T)\times\Omega, \qquad \omega_R|_{t=0} = \omega_0^R = \frac{\mathrm{curl} u_0}{a} \text{ in } \Omega$$

with

$$\mathrm{div}(au) = 0, \qquad \mathrm{curl}\, u = a\,\omega_R, \qquad au\cdot n|_{(0,T)\times\partial\Omega} = 0.$$

Remark. The boundary condition on au may be considered in a weak form if the boundary of the domain is not Lipschitz (Ω reduced by the support of a for example).

Definition. Let (u_0, ω_R^0) be such that

$$\mathrm{div}(au_0) = 0 \text{ in } \Omega, \qquad au_0\cdot n|_{\partial\Omega} = 0$$

and

$$\omega_R^0 \in L^\infty(\Omega), \qquad \mathrm{curl}\, u_0 = a\, \omega_R^0.$$

A couple (v, ω) is a global renormalized solution of the vorticity formulation of the lake equation with initial condition (v^0, ω^0) if

- $\omega_R \in L^\infty((0,T) \times \Omega)$ and $\sqrt{a} u \in L^\infty(0,T; L^2(\Omega))$
- $\mathrm{div}(au) = 0$ in $(0,T) \times \Omega$ and $a\, u \cdot n|_{(0,T)\times\partial\Omega} = 0$
- $\mathrm{curl}\, u = a\, \omega_R$ in the distributional sense.
- For all $\chi \in W^{1,\infty}(\mathbb{R})$ with $\chi(0) = 0$, choosing $\psi \in C_c^\infty(\Omega)$, then

$$\frac{d}{dt} \int_\Omega \chi(\omega_R)\, a\psi(x)\, dx - \int_\Omega \chi(\omega_R)\, au \cdot \nabla\psi\, dx = 0.$$

Using the stability process regarding the advective equation with anelastic constraint, we can get the following result

Theorem 7. *Let a be continuous on $\overline{\Omega}$ and strictly positive in Ω. Assume that $\nabla\sqrt{a} \in L^{2+}(\Omega)$ and that there exists $\eta > 0$ such that $1/a^\eta \in L^1(\Omega)$. Then there exists a global renormalized solution of the vorticity formulation of the lake Equation.*

Constructing an approximate sequence of global renormalized solution in the sense of the definition given above for $(a_\varepsilon, \alpha_\varepsilon)$ constructed in the paper and in the whole space $\mathbb{R}^2$ is an standard procedure since the coefficients and the domain are regular and the approximate bathymetry is far from vacuum, see for instance [27], [25], [8]. We get the following bounds uniform with respect to the parameter ε

$$u_\varepsilon \in L^\infty(0,T; L_a^2(\mathbb{R}^2)), \qquad \omega_R^\varepsilon \in L^\infty((0,T) \times \mathbb{R}^2). \tag{29}$$

Remark now that

$$\mathrm{curl}(a_\varepsilon u_\varepsilon) = u_\varepsilon \cdot \nabla^\perp a_\varepsilon + a_\varepsilon\, \mathrm{curl} u_\varepsilon = \sqrt{a_\varepsilon} u_\varepsilon \cdot \nabla^\perp \sqrt{a}_\varepsilon + a_\varepsilon^2\, \omega_R^\varepsilon$$

and

$$\mathrm{div}(a_\varepsilon u_\varepsilon) = 0.$$

This is the system that we will use to get regularity on $a_\varepsilon \nabla u_\varepsilon$ required in the hypothesis for the stability. Using the uniform bounds on ω_R^ε and $\sqrt{a}_\varepsilon u_\varepsilon$ and the uniform bound $\nabla\sqrt{a}_\varepsilon \in L^{2^+}(\Omega)$, we get that

$$a_\varepsilon u_\varepsilon \in L^\infty(0,T; W^{1,p}(\Omega)) \text{ for some } p > 1$$

Thus writing

$$a_\varepsilon \nabla u_\varepsilon = \nabla(a_\varepsilon u_\varepsilon) - u_\varepsilon \cdot \nabla a_\varepsilon$$

we get that, uniformly in ε,

$$a_\varepsilon \nabla u_\varepsilon \in L^\infty(0,T; L^p(\Omega)) \text{ for some } p > 1. \tag{30}$$

On the other hand,

$$\int_\Omega a_\varepsilon\, |\nabla u_\varepsilon|\, |\log a_\varepsilon|\, dx \le \frac{1}{\eta} \int_\Omega a_\varepsilon |\nabla u_\varepsilon| (\log(e + a_\varepsilon |\nabla u_\varepsilon|) - \log\eta) + \int_\Omega \frac{1}{a_\varepsilon^\eta}.$$

for $\eta > 0$ chosen such that $1/a_\varepsilon^\eta \in L^1(\Omega)$. By combining this with (30), we obtain a uniform bound on

$$\int_0^T \int_\Omega a_\varepsilon\, |\nabla u_\varepsilon|\, \log(e + |\nabla u_\varepsilon|)\, dx\, dt,$$

leading to the uniform bound on u_ε for the quantity $\|u\|_{a_\varepsilon}$ recalling that we already control u_ε uniformly in $L^\infty(0,T;L^2_a(\Omega))$. This allows then to use the stability procedure taking $\alpha = a_\varepsilon^k$ for any $k \geq 1$ to get the conclusion of the Theorem.

Remark. It is interesting to note that we get global renormalized solution instead of global weak solution as in [25]. In our result we use compactness on the vorticity through quantitative regularity estimate compared to compactness on the velocity field through the stream function equation and Aubin-Lions Lemma as usually.

Remark. Let us observe that assuming a behaves $\mathrm{dist}(x,\partial\Omega)^k$ the first hypothesis in the theorem asks for $k > 1$. The second hypothesis being satisfied. Of course we can generalize for more general power k playing with parameters θ using for instance that

$$a^\theta u \in L^\infty(0,T;W^{1,p}(\Omega))$$

and also

$$a^\theta \nabla u \in L^\infty(0,T;L^p(\Omega)) \text{ for some } p > 1$$

if $a^{\theta+1/2} \in L^{2+}(\Omega)$ and $\nabla a^{\theta-1/2} \in L^{2+}(\Omega)$.

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