EULER SPRAYS AND WASSERSTEIN GEOMETRY OF THE SPACE OF SHAPES

JIAN-GUO LIU

Department of Physics and Department of Mathematics
Duke University, Durham, NC 27708, USA

ROBERT L. PEGO AND DEJAN SLEPČEV

Department of Mathematical Sciences and Center for Nonlinear Analysis
Carnegie Mellon University, Pittsburgh, Pennsylvania, PA 12513, USA

Abstract. We study a distance between shapes defined by minimizing the integral of kinetic energy along transport paths constrained to measures with characteristic-function densities. The formal geodesic equations for this shape distance are Euler equations for incompressible, inviscid potential flow of fluid with zero pressure and surface tension on the free boundary. The minimization problem exhibits an instability associated with microdroplet formation, with the following outcomes: Shape distance is equal to Wasserstein distance. Furthermore, any two shapes of equal volume can be approximately connected by an Euler spray—a countable superposition of ellipsoidal droplet solutions of incompressible Euler equations. Every Wasserstein geodesic between shape densities is a weak limit of Euler sprays. Each Wasserstein geodesic is also the unique minimizer of a relaxed least-action principle for a fluid-vacuum mixture.

1. Introduction

In general, the problem of finding good ways to compare two signals (such as time series, images, or shapes) is important in a number of application areas, including computer vision, machine learning, and computational anatomy. Methods which endow the space of signals with the metric structure of a Riemannian manifold are of particular interest, as they facilitate a variety of image processing tasks. This geometric viewpoint, pioneered by Dupuis, Grenander & Miller [20, 26], Trouvé [44], Younes [50] and collaborators, has motivated the study of a variety of metrics on spaces of shapes and images over a number of years [13, 20, 25, 28, 34, 35, 41, 51, 52].

In a related development, distances derived from optimal transport theory (known as Monge-Kantorovich, Wasserstein, or earth-mover's distance) have been found useful in analyzing images [23, 27, 38, 42, 47, 48]. The transport distance with quadratic cost (Wasserstein

E-mail addresses: jliu@phy.duke.edu, rpego@cmu.edu, slepcev@math.cmu.edu.
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distance) is special as it provides a (formal) Riemannian structure on spaces of measures with fixed total mass [2, 37, 45].

In this paper we regard shapes as arbitrary *bounded measurable sets* in $\mathbb{R}^d$. To each shape $\Omega \subset \mathbb{R}^d$ we associate a measure in a natural way, namely the one whose density is the characteristic function $\mathbb{1}_\Omega$ of the shape. The Wasserstein distance between two such measures induces a distance between corresponding shapes of equal volume. But this induced distance does not immediately yield an induced notion of Wasserstein geometry, due to the fact that measures along Wasserstein geodesic paths typically do not have characteristic-functions densities, and thus do not correspond to shapes.

Hence we find it natural to investigate the geometry of a "submanifold" of the Wasserstein space consisting of measures corresponding to shapes. A similar idea was proposed recently by Schmitzer and Schnörr [41], who discussed restricting the Wasserstein metric to smooth paths of shape measures consisting of uniform distributions on bounded open sets in $\mathbb{R}^2$ with connected smooth boundary. In our present investigation, the only smoothness properties of shapes and paths that we require are those intrinsically associated with Wasserstein distance. We restrict our attention to paths of shapes of fixed volume in order to focus on morphology change and due to interesting relations of such paths to incompressible fluid flow. We return to indicate how our results apply to shape distance in the sense of [41] in the Extensions section at the end of this paper.

To be precise, we consider a distance between two shapes Ω_0 and Ω_1 (bounded measurable sets in $\mathbb{R}^d$) of equal volume, defined by minimizing an action that measures a cost for deforming one shape into the other:

$$d_s(\Omega_0, \Omega_1)^2 = \inf \mathcal{A}\,, \qquad \mathcal{A} = \int_0^1 \int_{\mathbb{R}^d} \rho |v|^2 \, dx \, dt\,, \tag{1.1}$$

where $\rho = (\rho_t)_{t\in[0,1]}$ is a path of *shape densities* transported by a velocity field $v \in L^2(\rho \, dx \, dt)$ according to the continuity equation

$$\partial_t \rho + \nabla \cdot (\rho v) = 0\,, \tag{1.2}$$

with the endpoint conditions

$$\rho_0 = \mathbb{1}_{\Omega_0}, \quad \rho_1 = \mathbb{1}_{\Omega_1}\,. \tag{1.3}$$

Here, saying that ρ_t is a shape density means that ρ_t is constrained to be a characteristic function for a shape Ω_t:

$$\rho_t = \mathbb{1}_{\Omega_t}\,, \quad t \in [0,1]. \tag{1.4}$$

Naturally, then, the velocity field must be divergence free in the interior of Ω_t, satisfying $\nabla \cdot v = 0$ there. Equation (1.2) holds in the sense of distributions in $\mathbb{R}^d \times [0,1]$, interpreting ρv as 0 wherever $\rho = 0$.

Let us write $d_W(\mathbb{1}_{\Omega_0}, \mathbb{1}_{\Omega_1})$ to denote the usual Wasserstein distance (Monge-Kanotorvich distance with quadratic cost) between the measures with densities $\mathbb{1}_{\Omega_0}$ and $\mathbb{1}_{\Omega_1}$. By the well-known result of Benamou and Brenier [4], $d_W(\mathbb{1}_{\Omega_0}, \mathbb{1}_{\Omega_1})^2$ is characterized as the infimum in (1.1) subject to the same transport and endpoint constraints as in (1.2)–(1.3), but *without* the constraint (1.4) that makes ρ a characteristic function. One expects that by restricting attention to paths of shape densities, the infimum in (1.1)-(1.4) should typically be larger—thus it is clear that

$$d_s(\Omega_0, \Omega_1) \geq d_W(\mathbb{1}_{\Omega_0}, \mathbb{1}_{\Omega_1}). \tag{1.5}$$

The first objective of the present work is to show that the volume-constrained optimal transport problem in (1.1)-(1.4) is subject to an instability associated with *microdroplet* formation. The infimum is typically not attained, and the value of the infimum itself is *the same* with or without the characteristic-function constraint. That is, the infimum yields squared Wasserstein distance unchanged:

Theorem 1.1. *For every pair of shapes (bounded measurable sets) in $\mathbb{R}^d$ of equal volume,*

$$d_s(\Omega_0, \Omega_1) = d_W(\mathbb{1}_{\Omega_0}, \mathbb{1}_{\Omega_1}).$$

The proof of this theorem, which we carry out in section 5, proceeds first in the case when both Ω_0 and Ω_1 are open sets. We decompose the source domain Ω_0, up to a set of measure zero, as a countable union of tiny disjoint open balls using a Vitali covering lemma. These 'microdroplets' are transported by a velocity field that is divergence-free and close to constant on each component. The droplets remain disjoint, and the total action or cost along the resulting path of 'spray' shape densities is then shown to be close to that attained by the displacement interpolant of the Monge-Kantorovich distance, which produces straight-line transport of points from the source Ω_0 to the target Ω_1. Figure 1 illustrates the result of a computation that illustrates these ideas. 'Microdroplet' subdomains of the source disk Ω_0 are transported by an incompressible flow to reach targets inside the hour-glass shape Ω_1 determined by the Brenier optimal transport map as described in section 2. (The Brenier map was computed using a method from [36]. In this example, it appears to be discontinuous along two line segments in Ω_0.)

(a) Source disk Ω_0 decomposed into microdroplets at $t = 0$.

(b) Microdroplets at $t = \frac{1}{2}$. Gray background is the support of the midpoint of Wasserstein geodesic from $\mathbb{1}_{\Omega_0}$ to $\mathbb{1}_{\Omega_1}$.

(c) Target shape Ω_1 with microdroplet images at $t = 1$.

FIGURE 1. Illustration of microdroplet volume-conserving flow from Ω_0 to Ω_1. Source Ω_0 is decomposed into countably many small balls, few of which are shown. Matching shades indicate corresponding droplets transported by flow. For $t \in (0, 1)$, droplets are contained in the linear interpolant of their source and target, and remain disjoint.

It is natural to ask next about the existence of *geodesic* paths connecting source to target. As it turns out (see Remarks 2.2–2.4), usually there is *no* length-minimizing path of shape densities for the problem (1.1)-(1.4), except in dimension $d = 1$, as a consequence of the uniqueness of the displacement interpolant as providing minimizing geodesic paths (action minimizers) for Monge-Kantorovich distance.

Nevertheless, it is interesting to study what targets can be reached from the source by following formal geodesics, which may not be length-minimizers but are critical paths for the variational problem in (1.1)–(1.4). This volume-constrained least-action principle is reminiscent of the ideas of V. I. Arnold that tie smooth paths of volume-preserving diffeomorphisms to incompressible fluid flow. In light of this connection, it is not surprising that the formal equations for geodesics of (1.1)–(1.4) should correspond to fluid equations of some kind.

As we show in section 3 below, it turns out that these geodesic equations are precisely the Euler equations for *incompressible, inviscid, potential flow* of fluid occupying domain Ω_t, with *zero surface tension and zero pressure* on the free boundary $\partial\Omega_t$. In short, the geodesic equations are classic water wave equations with zero gravity and surface tension. The initial-value problem for these equations has recently been studied in detail—the works [31, 15, 16] establish short-time existence and uniqueness for sufficiently smooth initial data.

A particular, simple solution will play a special role in our analysis. Namely, we observe (see Proposition 3.4) that a path $t \mapsto \Omega_t$ of ellipsoids is a critical point of the constrained action if and only if the d-dimensional vector $a(t) = (a_1(t), \dots, a_d(t))$, formed by the principal axis lengths, follows a geodesic curve on the hyperboloid-like surface in $\mathbb{R}^d$ determined by the constraint that corresponds to constant volume,

$$a_1 a_2 \cdots a_d = \text{const.} \tag{1.6}$$

While the question of determining which targets and sources are connected is difficult to answer in general, we find that for open sets, there exist solutions comprised of microdroplets (which we call *Euler sprays*) that approximately reach an arbitrary target as closely as desired.

Theorem 1.2. *Let Ω_0, Ω_1 be a pair of bounded open sets in $\mathbb{R}^d$ with equal volume. For any $\varepsilon > 0$, there is an Euler spray which transports the source Ω_0 (up to a null set) to a target Ω_1^ε whose L^∞ transportation distance from Ω_1 is less than ε. The action $\mathcal{A}^\varepsilon$ of the spray satisfies*

$$d_s(\Omega_0, \Omega_1^\varepsilon)^2 \le \mathcal{A}^\varepsilon \le d_W(\mathbb{1}_{\Omega_0}, \mathbb{1}_{\Omega_1})^2 + \varepsilon\,.$$

The precise definition of an Euler spray and the proof of this result will be provided in section 4. However, it is significant that the Euler sprays given by this theorem provide a family of weak solutions $(\rho^\varepsilon, v^\varepsilon, p^\varepsilon)$ to the following Euler system:

$$\partial_t \rho + \nabla\cdot(\rho v) = 0, \tag{1.7}$$

$$\partial_t(\rho v) + \nabla\cdot(\rho v \otimes v) + \nabla p = 0, \tag{1.8}$$

with the constraint that ρ^ε is a shape density, meaning it is a characteristic function as in (1.4). Both of these equations hold in the sense of distributions on $\mathbb{R}^d \times [0,1]$, which means the following: For any smooth test functions $q \in C_c^\infty(\mathbb{R}^d \times [0,1], \mathbb{R})$ and $\tilde v \in C_c^\infty(\mathbb{R}^d \times [0,1], \mathbb{R}^d)$,

$$\int_0^1 \int_{\mathbb{R}^d} \rho(\partial_t q + v\cdot\nabla q)\,dx\,dt = \int_{\mathbb{R}^d} \rho q\,dx\bigg|_{t=0}^{t=1}, \tag{1.9}$$

$$\int_0^1 \int_{\mathbb{R}^d} \rho v\cdot(\partial_t \tilde v + v\cdot\nabla\tilde v) + p\nabla\cdot\tilde v\,dx\,dt = \int_{\mathbb{R}^d} \rho v\cdot\tilde v\,dx\bigg|_{t=0}^{t=1}. \tag{1.10}$$

The limit as $\varepsilon \to 0$ for the sprays constructed in Theorem 1.2 can be characterized in terms of Wasserstein geodesics, as follows.

Theorem 1.3. *As $\varepsilon \to 0$, the weak solutions $(\rho^\varepsilon, v^\varepsilon, p^\varepsilon)$ associated to the Euler sprays of Theorem 1.2 converge to $(\rho, v, 0)$, where (ρ, v) is the weak solution*

$$\partial_t \rho + \nabla \cdot (\rho v) = 0, \tag{1.11}$$

$$\partial_t (\rho v) + \nabla \cdot (\rho v \otimes v) = 0, \tag{1.12}$$

provided by the Wasserstein geodesic (displacement interpolant) that connects the uniform measures on Ω_0 and Ω_1 as described in section 2. The convergence holds in the the following sense: $p^\varepsilon \to 0$ uniformly, and

$$\rho^\varepsilon \overset{\star}{\rightharpoonup} \rho, \qquad \rho^\varepsilon v^\varepsilon \overset{\star}{\rightharpoonup} \rho v, \qquad \rho^\varepsilon v^\varepsilon \otimes v^\varepsilon \overset{\star}{\rightharpoonup} \rho v \otimes v, \tag{1.13}$$

weak-$\star$ in L^∞ on $\mathbb{R}^d \times [0,1]$.

The convergence in (1.13) can be strengthened in terms of the TL^p topology introduced in [24] to compare two functions that are absolutely continuous with respect to different probability measures—see Remark 4.7.

One further striking connection between Wasserstein geodesics and least-action principles for incompressible fluid flow will be developed in this paper. In particular this relates to work of Brenier on relaxations of Arnold's least-action principle for incompressible flow [5, 7, 8, 9, 10, 11]. We will describe a relaxed least-action principle for incompressible flow of two-fluid mixtures that is a variant of Brenier's model for homogenized vortex sheets [8], and is related to the variable-density model studied by Lopes et al. [32]. Our model, however, also allows one fluid to have zero density, corresponding to a fluid-vacuum mixture. In this degenerate case, we show that the Wasserstein geodesic provides the unique minimizer of the relaxed least-action principle—see Theorem 6.2. Moreover, the smooth sprays constructed in Theorem 5.2 provide a minimizing sequence consisting of unmixed paths—paths of shape densities.

The plan of this paper is as follows. In section 2 we collect some basic facts and estimates that concern geodesics for Monge-Kantorovich/Wasserstein distance. In section 3 we derive formally the geodesic equations for paths of shape densities and describe the special class of ellipsoidal solutions. The construction of Euler sprays and the proof of Theorem 1.2 is carried out in section 4. Theorem 1.1 is proved in section 5. The connection between Wasserstein geodesics and relaxed least-action priniciple motivated by Brenier's work is developed in section 6.

The paper concludes in section 7 with a discussion of the notion of shape distance examined by Schmitzer and Schnörr in [41]. In particular, we extend the result of Theorem 1.1, for volume-constrained paths of shapes, to show that a shape distance determined by paths of uniform measures again agrees with the Wasserstein distance between the endpoints. The proof involves a displacement convexity argument that makes use of the well-known fact that $\rho^{-1/d}$ is concave along particle paths of Wasserstein geodesics.

1.1. **Related work on the geometry of image and shape space.** The idea to use deformations as a means of comparing images goes back to pioneering work of D'Arcy Thompson [43]. Dupuis, Grenander, Miller [20, 26], Trouvé [44], and Younes [50] introduced the concepts of differential geometry to study spaces of images and shapes. The main thrust of these works is to study Riemannian metrics and the resulting distances in the space of image and shape deformations. Connections with Arnold viewpoint of fluid mechanics were noted from the outset [50], and have been further explored by Holm, Trouvé, Younes and others [25, 28, 51]. This work has led to the *Euler-Poincaré theory of metamorphosis* [28], which sets up a formalism for analyzing least-action principles based on Lie-group symmetries generated by diffeomorphism groups.

To obtain regularity of the minimizing paths and resulting diffeomorphisms, the Riemannian metrics considered often penalize the integrals of second-order derivatives of velocities, as in the Large Deformation Diffeomorphic Metric Mapping (LDDMM) approach of [3]. Metrics based on conservative transport which penalize only one derivative of the velocity field are connected with viscous dissipation in fluids and have been considered by Rumpf, Wirth and collaborators [39, 49], as well as by Brenier, Otto, and Seis [12], who established a connection to optimal transport. As we mentioned at the beginning, metrics which penalize only L^2 norm of the velocity have strong connections to optimal transportation.

A different way to consider shapes is to study them only via their boundary, and consider metrics which are based on penalizing normal velocity of the boundary. Such a point of view has been taken by Michor, Mumford and collaborators [13, 34, 35, 52]. As they show in [34], penalizing only the L^2 norm of normal velocity does not lead to a viable geometry, as any two states can be connected by an arbitrarily short curve. On the other hand it is shown in [13] that if two or more derivatives of the normal velocity are penalized, then the resulting metric on the shape space is geodesically complete.

In this context, we note that what our work shows is that if the L^2 norm of the transport velocity is considered in the bulk, then the global metric distance is not zero, but that it is still degenerate in the sense that a length-minimizing geodesic does not exist in the shape space. We speculate that to create a shape distance that (even locally) admits length-minimizing paths in the space of shapes, one needs to prevent the creation a large perimeter at negligible cost. This is somewhat analogous to the motivation for the metrics on the space of curves considered by Michor and Mumford [34]. Possibilities include introducing a term in the metric which penalizes deforming the boundary, or a term which enforces greater regularity for the vector fields considered.

2. Preliminaries: Wasserstein geodesics between shapes

In this section we recall some basic properties of the standard minimizing geodesic paths (displacement interpolants) for the Wasserstein or Monge-Kantorovich distance between shape densities, and establish some basic estimates. A property that is key in the sequel is that the density ρ is *convex* along the corresponding particle paths, see Lemma 2.1.

2.1. **Standard Wasserstein geodesics.**

Let Ω_0 and Ω_1 be two shapes in $\mathbb{R}^d$ (bounded open sets) with equal volume. Let μ_0 and μ_1 be measures with densities $\rho_0 = \mathbb{1}_{\Omega_0}$ and $\rho_1 = \mathbb{1}_{\Omega_1}$, respectively. As is well known [6, 30], there exists a convex function ψ such that $T = \nabla\psi$ (called the *Brenier map* in [45]) is the optimal transportation map between Ω_0 and Ω_1 corresponding to the quadratic cost. Moreover, this map is unique a.e. in Ω_0; see [6] or [45, Thm. 2.32].

McCann [33] later introduced a natural curve $t \mapsto \mu_t$ that interpolates between μ_0 and μ_1, called the *displacement interpolant*, which can be described as the push-forward of the measure μ_0 by the interpolation map T_t given by

$$T_t(z) = (1-t)z + t\nabla\psi(z), \qquad 0 \le t \le 1. \tag{2.1}$$

Note that particle paths $z \mapsto T_t(z)$ follow straight lines with constant velocity:

$$v(T_t(z), t) = \nabla\psi(z) - z. \tag{2.2}$$

Furthermore μ_t has density ρ_t that satisfies the continuity equation

$$\partial_t \rho + \operatorname{div}(\rho v) = 0. \tag{2.3}$$

In terms of these quantities, the Wasserstein distance satisfies

$$d_W(\mu_0, \mu_1) = \int_{\Omega_0} |\nabla\psi(z) - z|^2 \, dz = \int_0^1 \int_{\Omega_t} \rho |v|^2 \, dx \, dt \ ,$$

and the displacement interpolant has the property that

$$d_W(\mu_s, \mu_t) = (t-s) d_W(\mu_0, \mu_1), \quad 0 \le s \le t \le 1. \tag{2.4}$$

The property (2.4) implies that the displacement interpolant is a *constant-speed geodesic* (length-minimizing path) with respect to Wasserstein distance. The displacement interpolant $t \mapsto \mu_t$ is the *unique* constant-speed geodesic connecting μ_0 and μ_1, due to the uniqueness of the Brenier map and Proposition 5.32 of [40] (or see [1, Thm. 3.10]]). For brevity the path $t \mapsto \mu_t$ is called the *Wasserstein geodesic* from μ_0 to μ_1.

Extending the regularity theory of Caffarelli [14], Figalli [21] and Figalli & Kim [22] have shown (see Theorem 3.4 in [17] and also [18]) that the optimal transportation potential ψ is smooth away form a set of measure zero. More precisely, there exist relatively closed sets of measure zero, $\Sigma_i \subset \Omega_i$ for $i = 0, 1$ such that $T : \Omega_0 \backslash \Sigma_0 \to \Omega_1 \backslash \Sigma_1$ is a smooth diffeomorphism between two open sets.

Let $\lambda_1(z), \ldots, \lambda_d(z)$ be the eigenvalues of $\operatorname{Hess} \psi(z)$ for $z \in \Omega_0 \backslash \Sigma_0$. Due to convexity and regularity of ψ, $\lambda_i > 0$ for all $i = 1, \ldots, n$. Furthermore, because $\nabla\psi$ is a map that pushes forward the Lebesgue measure on Ω_0 to that on Ω_1, it follows that the Jacobian of T has value 1 and thus $\lambda_1 \cdots \lambda_d = 1$.

Along the particle paths of displacement interpolation starting from any $z \in \Omega_0 \setminus \Sigma_0$, the mass density satisfies

$$\rho(T_t(z), t)^{-1} = \det \frac{\partial T_t}{\partial z} = \det((1-t)I + t\nabla^2\psi(z)) = \prod_{j=1}^{d} (1 - t + t\lambda_j(z)) \tag{2.5}$$

We now show that the density ρ is convex along these paths. The stronger fact that $\rho^{-1/d}$ is concave along particle paths follows from more general classical results stated in [33] and related to a well-known proof of the Brunn-Minkowski inequality by Hadwiger and Ohmann. Since a simple proof is available for our case, we present it here for completeness.

Lemma 2.1. *Along the particle paths $t \mapsto T_t(z)$ of displacement interpolation between the measures μ_0 and μ_1 with respective densities $\mathbb{1}_{\Omega_0}$ and Ω_1 as above, the map $t \mapsto \rho(T_t(z), t)^{-1/d}$ is concave. Further, the map $t \mapsto \rho(T_t(z), t)$ is convex. Moreover, $\rho \le 1$.*

Proof. Fix z and let $g(t) = \rho(T_t(z), t)^{-1/d}$. We compute

$$\frac{g'}{g} = \frac{1}{d} \sum_{j=1}^{d} \frac{\lambda_j - 1}{1 - t + t\lambda_j},$$

$$\frac{g''}{g} = \left(\frac{1}{d} \sum_{j=1}^{d} \frac{\lambda_j - 1}{1 - t + t\lambda_j} \right)^2 - \frac{1}{d} \sum_{j=1}^{d} \left(\frac{\lambda_j - 1}{1 - t + t\lambda_j} \right)^2 \le 0 \tag{2.6}$$

due to the Cauchy-Schwartz (or Jensen's) inequality. This shows g is concave. That $t \mapsto \rho(T_t(z), t)$ is convex follows directly. Because ρ equals 1 when $t = 0$ and $t = 1$, we infer $\rho \le 1$ along particle paths. □

We also note that computations above and continuity equation (2.2) imply

$$\operatorname{div} v = -\frac{1}{\rho}\left(\frac{d\rho}{dt}\right) = -\frac{d}{dt}\log\rho = \sum_{j=1}^{d}\frac{\lambda_j - 1}{1 - t + t\lambda_j}. \tag{2.7}$$

Another well-known fact about the Eulerian velocity that we will use (in Lemma 5.5) is that $v(\cdot,t)$ is a spatial gradient for $t \in (0,1)$. Namely, note $T_t = \nabla\psi_t$ where $\psi_t(z) = \frac{1}{2}(1-t)|z|^2 + t\psi(z)$ is strictly convex, with Legendre transform ψ_t^* which satisfies

$$\psi_t^*(\nabla\psi_t(z)) = \langle z, \nabla\psi_t(z)\rangle - \psi_t(z), \qquad \nabla\psi_t^* \circ \nabla\psi_t(z) = z. \tag{2.8}$$

(The latter identity is a classical fact easily checked by differentiation for z in the nonsingular set.) Then by combining this with (2.1)–(2.2) we find (for $x = \nabla\psi_t(z)$)

$$\nabla\psi_t^*(x) + tv(x,t) = x = \nabla|x|^2/2. \tag{2.9}$$

As an alternative expression, one can check that

$$v(x,t) = \nabla\phi_t(x), \qquad \phi_t(x) = \psi(z) - z + \frac{t}{2}|\nabla\psi(z) - z|^2. \tag{2.10}$$

Remark 2.2. It is interesting to ask when it is possible that $\rho(T_t(z),t) \equiv 1$ for all z in the non-singular set $\Omega_0 \setminus \Sigma_0$, for this is the case if and only if there exists some action minimizing path of shape densities for the problem (1.1)–(1.4). (To establish the equivalence, one shows that necessarily $\Omega_t = T_t(\Omega_0 \setminus \Sigma_0)$ up to a set of measure zero, by invoking the uniqueness of the Wasserstein geodesic as discussed above. For this to hold, clearly it is a necessary consequence of (2.6) that $\lambda_j \equiv 1$ everywhere in $\Omega_0 \setminus \Sigma_0$. This means T is a rigid translation on each component of $\Omega_0 \setminus \Sigma_0$.

Remark 2.3. As a nontrivial example in the case of one dimension ($d = 1$), let $\mathcal{C} \subset [0,1]$ be the standard Cantor set, and let $\Omega_0 = (0,1)$. Define the Brenier map $T(x) = x + c(x)$ with c given by the Cantor function, increasing and continuous on $[0,1]$ with $c(0) = 0$, $c(1) = 1$ and $c' = 0$ on $(0,1) \setminus \mathcal{C}$. Then $T(\Omega_0) = (0,2)$, but the pushforward of uniform measure on Ω_0 is the uniform measure on the set $\Omega_1 = T(\Omega_0 \setminus \mathcal{C})$, which has countably many components, and total length $|\Omega_1| = 1$.

Remark 2.4. Actually, in the case $d = 1$ it is always the case that $\rho(T_t(z),t) \equiv 1$ for all z in the non-singular set. This is so because the diffeomorphism $T : \Omega_0\backslash\Sigma_0 \to \Omega_1\backslash\Sigma_1$ must be a rigid translation on each component, as it pushes forward Lebesgue measure to Lebesgue measure.

2.2. **Local linear approximation and estimates.** Let $\lambda_1, \ldots, \lambda_d$ be the eigenvalues of $\operatorname{Hess}\psi(x)$, as before. Recall that $\lambda_i > 0$ for all $i = 1, \ldots, n$ and $\lambda_1 \cdots \lambda_d = 1$. Let $\underline{\lambda}(x)$ and $\overline{\lambda}(x)$ be the minimal and maximal eigenvalues of $\operatorname{Hess}\psi(x) = DT(x)$ respectively. We define, for any $U \subset \Omega_0\backslash\Sigma_0$,

$$\underline{\lambda}_U = \inf\{\underline{\lambda}(x) : x \in U\}, \qquad \overline{\lambda}_U = \sup\{\overline{\lambda}(x) : x \in U\}, \tag{2.11}$$

and note that for any $x \in U$ and $\hat{x} \in \mathbb{R}^d$,

$$\underline{\lambda}_U|\hat{x}| \le |DT(x)\hat{x}| \le \overline{\lambda}_U|\hat{x}|. \tag{2.12}$$

For $U \Subset \Omega_0\backslash\Sigma_0$ we also let

$$\|D^3\psi\|_U := \sup_{x\in U}\max_{|u|=|v|=|w|=1}\left|\sum_{i,j,k=1}^{d}\frac{\partial^3\psi(x)}{\partial x_i\partial x_j\partial x_k}\,u_iv_jw_k\right|. \tag{2.13}$$

Taylor expansion provides a basic estimate on the difference between the optimal transport map and its linearization: Whenever $B(x_0,r)\subset\Omega_0\setminus\Sigma_0$ and $x\in B(x_0,r)$,

$$|T(x)-T(x_0)-DT(x_0)(x-x_0)|<\frac{1}{2}\|D^3\psi\|_{B(x_0,r)}\,r^2. \tag{2.14}$$

3. Geodesics and incompressible fluid flow

3.1. **Incompressible Euler equations for smooth critical paths.** In this subsection, for completeness we derive the Euler fluid equations that formally describe smooth geodesics (paths with stationary action) for the shape distance in (1.1)-(1.4). To cope with the problem of moving domains we work in a Lagrangian framework, computing variations with respect to flow maps that preserve density and the endpoint shapes Ω_0 and Ω_1.

Toward this end, suppose that

$$Q=\bigcup_{t\in[0,1]}\Omega_t\times\{t\}\quad\subset\mathbb{R}^d\times[0,1] \tag{3.1}$$

is a space-time domain generated by smooth deformation of Ω_0 due to a smooth velocity field $v\colon\bar{Q}\to\mathbb{R}^d$. That is, the t-cross section of Q is given by

$$\boxed{\Omega_t=X(\Omega_0,t),} \tag{3.2}$$

where X is the Lagrangian flow map associated to v, satisfying

$$\dot{X}(z,t)=v(X(z,t),t),\qquad X(z,0)=z, \tag{3.3}$$

for all $(z,t)\in\Omega_0\times[0,1]$.

For any (smooth) extension of v to $\mathbb{R}^d\times[0,1]$, the solution of the mass-transport equation in (1.2) with given initial density ρ_0 supported in $\bar{\Omega}_0$ is

$$\rho(x,t)=\rho_0(z)\det\left(\frac{\partial X}{\partial z}(z,t)\right)^{-1},\quad x=X(z,t)\in\Omega_t,$$

with $\rho=0$ outside Q.

Considering a smooth family $X=X_\varepsilon$ of flow maps defined for all small values of a variational parameter ε, the variation $\delta X=(\partial X/\partial\varepsilon)|_{\varepsilon=0}$ induces a variation in density satisfying

$$-\frac{\delta\rho}{\rho}=\delta\log\det\left(\frac{\partial X}{\partial z}(z,t)\right)=\operatorname{tr}\left(\frac{\partial\delta X}{\partial z}\left(\frac{\partial X}{\partial z}\right)^{-1}\right) \tag{3.4}$$

Introducing $\tilde{v}(x,t)=\delta X(z,t)$, $x=X(z,t)$, it follows

$$-\frac{\delta\rho}{\rho}=\nabla\cdot\tilde{v}. \tag{3.5}$$

For variations that leave the density invariant, necessarily $\nabla\cdot\tilde{v}=0$.

We now turn to consider the variation of the action or transport cost as expressed in terms of the flow map:

$$\mathcal{A}=\int_0^1\int_{\mathbb{R}^d}\rho|v|^2dx\,dt=\int_0^1\int_{\Omega_0}|\dot{X}(z,t)|^2dz\,dt\,. \tag{3.6}$$

For flows preserving $\rho = 1$ in Q, of course $\nabla \cdot v = 0$. Computing the first variation of $\mathcal{A}$ about such a flow, after an integration by parts in t and changing to Eulerian variables, we find

$$
\begin{aligned}
\frac{\delta \mathcal{A}}{2} &= \int_0^1 \int_{\Omega_0} \dot{X} \cdot \delta \dot{X} \, dz \, dt \\
&= \int_{\Omega_0} \dot{X} \cdot \delta X \, dz \bigg|_{t=1} - \int_0^1 \int_{\Omega_0} \ddot{X} \cdot \delta X \, dz \, dt \\
&= \int_{\Omega_t} v \cdot \tilde{v} \, dx \bigg|_{t=1} - \int_0^1 \int_{\Omega_t} (\partial_t v + v \cdot \nabla v) \cdot \tilde{v} \, dx \, dt. \qquad (3.7)
\end{aligned}
$$

Recall that any L^2 vector field u on Ω_t has a unique Helmholtz decomposition as the sum of a gradient and a field L^2-orthogonal to all gradients, which is divergence-free with zero normal component at the boundary:

$$
u = \nabla p + w, \qquad \nabla \cdot w = 0 \ \text{ in } \Omega_t, \quad w \cdot n = 0 \ \text{ on } \partial\Omega_t. \tag{3.8}
$$

If we loosen the requirement that $w \cdot n = 0$ on the boundary, it is still the case that

$$
\int_{\partial\Omega_t} w \cdot n \, dS = \int_{\Omega_t} \nabla \cdot w \, dx = 0,
$$

It follows that the space orthogonal to all divergence-free fields on Ω_t is the space of gradients ∇p such that p is constant on the boundary, and we may take this constant to be zero:

$$
\boxed{p = 0 \quad \text{on } \partial\Omega_t.} \tag{3.9}
$$

Requiring $\delta\mathcal{A} = 0$ for arbitrary virtual displacements having $\nabla \cdot \tilde{v} = 0$ (and $\tilde{v} = 0$ at $t = 1$ at first), we find that necessarily $u = -(\partial_t v + v \cdot \nabla v)$ is such a gradient. Thus the incompressible Euler equations hold in Q:

$$
\partial_t v + v \cdot \nabla v + \nabla p = 0\,, \quad \nabla \cdot v = 0 \quad \text{in } Q, \tag{3.10}
$$

where $p : \bar{Q} \to \mathbb{R}$ is smooth and satisfies (3.9).

Finally, we may consider variations $\tilde{v}$ that do not vanish at $t = 1$. However, we require $\tilde{v} \cdot n = 0$ on $\partial\Omega_1$ in this case because the target domain Ω_1 should be fixed. That is, the allowed variations in the flow map X must fix the image at $t = 1$:

$$
\Omega_1 = X(\Omega_0, 1). \tag{3.11}
$$

The vanishing of the integral term at $t = 1$ in (3.7) then leads to the requirement that v is a gradient at $t = 1$. For $t = 1$ we must have

$$
v = \nabla\phi \quad \text{in } \Omega_t. \tag{3.12}
$$

We claim this gradient representation actually must hold for all $t \in [0, 1]$. Let $v = \nabla\phi + w$ be the Helmholtz decomposition of v, and for small ε consider the family of flow maps generated by

$$
\dot{X}(z, t) = (v + \varepsilon w)(X(z, t), t) \qquad X(z, 0) = z. \tag{3.13}
$$

Corresponding to this family, the action from (3.6) takes the form

$$
\mathcal{A} = \int_0^1 \int_{\Omega_0} |\dot{X}(z, t)|^2 dz \, dt = \int_0^1 \int_{\Omega_t} |\nabla\phi|^2 + |(1 + \varepsilon)w|^2 \, dx \, dt \tag{3.14}
$$

Because $w \cdot n = 0$ on $\partial\Omega_t$, the domains Ω_t do not depend on ε, and the same is true of $\nabla\phi$ and w, so this expression is a simple quadratic polynomial in ε. Thus

$$\frac{1}{2}\left.\frac{d\mathcal{A}}{d\varepsilon}\right|_{\varepsilon=0} = \int_0^1 \int_{\Omega_t} |w|^2 \, dx \, dt \tag{3.15}$$

and consequently it is necessary that $w = 0$ if $\delta\mathcal{A} = 0$. This proves the claim.

The Euler equation in (3.10) is now a spatial gradient, and one can add a function of t alone to ϕ to ensure that

$$\boxed{\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + p = 0, \quad \Delta\phi = 0 \quad \text{in } \Omega_t.} \tag{3.16}$$

The equations boxed above, including (3.16) together with the zero-pressure boundary condition (3.9) and the kinematic condition that the boundary of Ω_t moves with normal velocity $v \cdot n$ (coming from (3.2)-(3.3)), comprise what we shall call the *Euler droplet* equations, for incompressible, inviscid, potential flow of fluid with zero surface tension and zero pressure at the boundary.

Definition 3.1. *A smooth* solution *of the Euler droplet equations is a triple* (Q, ϕ, p) *such that* $\phi, p\colon \bar{Q} \to \mathbb{R}$ *are smooth and the equations* (3.1), (3.2), (3.3), (3.12), (3.16), (3.9) *all hold.*

Proposition 3.2. *For smooth flows X that deform Ω_0 as above, that respect the density constraint $\rho = 1$ and fix $\Omega_1 = X(\Omega_0, 1)$, the action $\mathcal{A}$ in* (3.6) *is critical with respect to smooth variations if and only if X corresponds to a smooth solution of the Euler droplet equations.*

3.2. **Weak solutions and Galilean boost.** Here we record a couple of simple basic properties of solutions of the Euler droplet equations.

Proposition 3.3. *Let (Q, ϕ, p) be a smooth solution of the Euler droplet equations. Let $\rho = \mathbb{1}_Q$ and $v = \mathbb{1}_Q \nabla\phi$, and extend p as zero outside Q.*
(a) The Euler equations (1.7)-(1.8) *hold in the sense of distributions on* $\mathbb{R}^d \times [0, 1]$.
(b) The mean velocity

$$\bar{v} = \frac{1}{|\Omega_t|} \int_{\Omega_t} v(x, t) \, dx \tag{3.17}$$

is constant in time, and the action decomposes as

$$\mathcal{A} = \int_0^1 \int_{\Omega_t} |v - \bar{v}|^2 dx \, dt + |\Omega_0||\bar{v}|^2 \,. \tag{3.18}$$

(c) Given any constant vector $b \in \mathbb{R}^d$, another smooth solution $(\hat{Q}, \hat{\phi}, \hat{p})$ of the Euler droplet equations is given by a Galilean boost, via

$$\hat{Q} = \bigcup_{t \in [0,1]} (\Omega_t + bt) \times \{t\} \,, \tag{3.19}$$

$$\hat{\phi}(x + bt, t) = \phi(x, t) + b \cdot x + \frac{1}{2}|b|^2 t \,, \quad \hat{p}(x + bt, t) = p(x, t) \,. \tag{3.20}$$

Proof. To prove (a), what we must show is the following: For any smooth test functions $q \in C_c^\infty(\mathbb{R}^d \times [0,1], \mathbb{R})$ and $\tilde{v} \in C_c^\infty(\mathbb{R}^d \times [0,1], \mathbb{R}^d)$,

$$\int_Q (\partial_t q + v \cdot \nabla q)\, dx\, dt = \int_{\Omega_t} q\, dx \Big|_{t=0}^{t=1} \tag{3.21}$$

$$\int_Q v \cdot (\partial_t \tilde{v} + v \cdot \nabla \tilde{v}) + p \nabla \cdot \tilde{v}\, dx\, dt = \int_{\Omega_t} \tilde{v} \cdot v\, dx \Big|_{t=0}^{t=1} \tag{3.22}$$

Changing to Lagrangian variables via $x = X(z,t)$, writing $\hat{q}(z,t) = q(x,t)$, and using incompressibility, equation (3.21) is equivalent to

$$\int_0^1 \int_{\Omega_0} \frac{d}{dt} \hat{q}(z,t)\, dz\, dt = \int_{\Omega_0} \hat{q}(z,t)\, dz \Big|_{t=0}^{t=1}. \tag{3.23}$$

Evidently this holds. In (3.22), we integrate the pressure term by parts, and treat the rest as in (3.7) to find that (3.22) is equivalent to

$$\int_Q (\partial_t v + v \cdot \nabla v + \nabla p) \cdot \tilde{v}\, dx\, dt = 0. \tag{3.24}$$

Then (a) follows. The proof of parts (b) and (c) is straightforward. □

3.3. **Ellipsoidal Euler droplets.** The intial-value problem for the Euler droplet equations is a difficult fluid free boundary problem. For flows with vorticity and smooth enough intial data, smooth solutions for short time have been shown to exist in [31, 15, 16].

In this section, we describe simple, particular Euler droplet solutions for which the fluid domain Ω_t remains ellipsoidal for all t. Our main result is the following.

Proposition 3.4. *Given a constant $r > 0$, let $a(t) = (a_1(t), \dots, a_d(t))$ be any constant-speed geodesic on the surface in $\mathbb{R}^d_+$ determined by the relation*

$$a_1 \cdots a_d = r^d. \tag{3.25}$$

Then this determines an Euler droplet solution (Q, ϕ, p) with Ω_t equal to the ellipsoid $E_{a(t)}$ given by

$$E_a = \Big\{ x \in \mathbb{R}^d : \sum_j (x_j/a_j)^2 < 1 \Big\}, \tag{3.26}$$

and potential and pressure given by

$$\phi(x,t) = \frac{1}{2} \sum_j \frac{\dot{a}_j x_j^2}{a_j} - \beta(t)\,, \qquad p(x,t) = \dot{\beta} \left(1 - \sum_j \frac{x_j^2}{a_j^2} \right), \tag{3.27}$$

with

$$\dot{\beta}(t) = \frac{1}{2} \frac{\sum_j \dot{a}_j^2 / a_j^2}{\sum_j 1/a_j^2}. \tag{3.28}$$

For clarity, we first derive the result in the planar case, then treat the case of general dimension $d \geq 2$.

3.3.1. *Droplets in dimension* $d = 2$. We seek incompressible flows inside a time-dependent elliptical domain where

$$\frac{x^2}{a(t)^2} + \frac{y^2}{b(t)^2} < 1, \tag{3.29}$$

with the geometric mean $r = (ab)^{1/2}$ constant in time for volume conservation. We will find such flows as time-stretched straining flows (X, Y), satisfying

$$(\dot{X}, \dot{Y}) = v(X, Y, t) = \alpha(t)(X, -Y)\,.$$

Such flows have velocity potential satisfying $v = \nabla\phi$, with

$$\phi(x, y, t) = \frac{1}{2}\alpha(t)(x^2 - y^2) - \beta(t), \tag{3.30}$$

$$\partial_t\phi = \frac{1}{2}\dot{\alpha}(x^2 - y^2) - \dot{\beta}, \quad \frac{1}{2}|\nabla\phi|^2 = \frac{1}{2}\alpha^2(x^2 + y^2)\,.$$

To satisfy the Bernoulli equation we require $\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 = 0$ on the boundary of the ellipse $(x, y) = (a\cos\theta, b\sin\theta)$, or

$$(\dot{\alpha} + \alpha^2)a^2\cos^2\theta + (-\dot{\alpha} + \alpha^2)b^2\sin^2\theta = 2\dot{\beta}$$

In order for this to hold independent of θ, we require

$$(\dot{\alpha} + \alpha^2)a^2 = -(\dot{\alpha} - \alpha^2)b^2 = 2\dot{\beta}.$$

Due to the motion of the boundary points $(a, 0)$, $(0, b)$ we need

$$\dot{a} = \alpha a, \quad \dot{b} = -\alpha b,$$

whence

$$2\dot{\beta} = a\ddot{a} = \frac{2b^2\dot{a}^2}{(a^2 + b^2)} = \frac{2r^4\dot{a}^2}{(a^4 + r^4)}$$

because $r^2 = ab$ is constant. Notice $\ddot{a} > 0$ in all cases. There is a first integral (because kinetic energy is conserved) which we can find by writing

$$\frac{\ddot{a}}{\dot{a}} = 2\dot{a}\left(\frac{1}{a} - \frac{a^3}{r^4 + a^4}\right),$$

whence we find that $a(t)$ and $b(t)$ are determined by solving

$$\frac{\dot{a}}{a} = \frac{c}{\sqrt{a^2 + b^2}} = -\frac{\dot{b}}{b} = \alpha(t). \tag{3.31}$$

for some real constant c. From the derivation of the Bernoulli equation, inside the ellipse the pressure is

$$p = -\partial_t\phi - \frac{1}{2}|\nabla\phi|^2 = \dot{\beta}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)\,. \tag{3.32}$$

where $\dot{\beta}$ is recovered from the equation

$$\dot{\beta}(t) = \left(\frac{cab}{a^2 + b^2}\right)^2\,. \tag{3.33}$$

To summarize, an elliptical Euler droplet solution (Q, ϕ, p) is determined in terms of any solution $(a(t), b(t))$ of (3.31) (with any real c) by (3.29), (3.30), (3.32), and (3.33). We note that the speed of motion of the point (a, b) on the hyperbola $ab = r^2$ is constant: by (3.31),

$$\dot{a}^2 + \dot{b}^2 = c^2. \tag{3.34}$$

In the context of the fixed-endpoint problem, then, $|c|$ is the distance along the hyperbola betweeen $(a(0), b(0))$ and $(a(1), b(1))$.

3.3.2. *Droplets in dimension $d \geq 2$.* Let us now derive the result stated in Proposition 3.4. The flow X associated with a velocity potential of the form in (3.27) must satisfy

$$\dot{X}_j = \alpha_j(t) X_j, \qquad \alpha_j = \frac{\dot{a}_j}{a_j}, \quad j = 1, \ldots, d. \tag{3.35}$$

Then $(X_j/a_j)^{\cdot} = 0$ for all j, so the flow is purely dilational along each axis and consequently ellipsoids are deformed to ellipsoids as claimed. Note that incompressibility corresponds to the relation

$$\Delta\phi = \sum_j \alpha_j = \sum_j \frac{\dot{a}_j}{a_j} = \frac{d}{dt}\log(a_1 \cdots a_d) = 0.$$

From (3.27) we next compute

$$\partial_t \phi_t + \frac{1}{2}|\nabla\phi|^2 = -\dot{\beta} + \frac{1}{2}\sum_j (\dot{\alpha}_j + \alpha_j^2) x_j^2 = -\dot{\beta} + \frac{1}{2}\sum_j \frac{\ddot{a}_j x_j^2}{a_j}.$$

This must equal zero on the boundary where $x_j = a_j s_j$ with $s \in S_{d-1}$ arbitrary. We infer that for all j,

$$a_j \ddot{a}_j = 2\dot{\beta}. \tag{3.36}$$

The expression for pressure in (3.27) in terms of $\dot{\beta}$ then follows from (3.16), and $p = 0$ on $\partial\Omega_t$.

We recover $\dot{\beta}$ by differentiating the constraint twice in time. We find

$$\begin{aligned} 0 &= \sum_j \left(\sum_k a_1 \cdots a_d \frac{\dot{a}_k}{a_k}\frac{\dot{a}_j}{a_j} + a_1 \ldots a_d \frac{a_j \ddot{a}_j - \dot{a}_j^2}{a_j^2} \right) \\ &= 0 + \sum_j \frac{2\dot{\beta} - \dot{a}_j^2}{a_j^2} \end{aligned}$$

whence (3.28) holds.

To get the first integral that corresponds to kinetic energy, multiply (3.36) by $2\dot{a}_j/a_j$ and sum to find

$$0 = \sum_j \dot{a}_j \ddot{a}_j, \quad \text{whence} \quad \sum_j \dot{a}_j^2 = c^2$$

and we see that $c = |\dot{a}(t)|$ is the constant speed of motion.

It remains to see that (3.36) are the geodesic equations on the constraint surface. To see this, recall that geodesic flow on the constraint surface corresponds to a stationary point for the augmented action

$$\int_0^1 \frac{1}{2}|\dot{a}|^2 + \lambda(t)\left(\prod_j a_j - r^d\right)\, dt$$

which leads to the Euler-Lagrange equations

$$-\ddot{a}_j + \frac{\lambda(t) r^d}{a_j} = 0.$$

Correspondingly, $\lambda r^d = 2\dot{\beta}$. This finishes the demonstration of Proposition 3.4.

Remark 3.5. For later reference, we note that $\ddot{a}_j > 0$ for all t, due to (3.36) and (3.28).

Remark 3.6. Given any two points on the surface described by the constraint (3.25), there exists a constant-speed geodesic connecting them. This fact is a straightforward consequence of the Hopf-Rinow theorem [29, Theorem 1.7.1], because all closed and bounded subsets on the surface are compact.

Remark 3.7. The Euclidean metric on the hyperboloid-like surface arises, in fact, as the metric induced by the Wasserstein distance [46, Chap. 15], because, given any dilational flow satisfying (3.35) with $a_1 \cdots a_d = r^d$,

$$\int_{\Omega_t} |v|^2 \, dx = \int_{\Omega_t} \sum_j \alpha_j^2 x_j^2 \, dx = \sum_j \dot{a}_j^2 \int_{|z| \le 1} z_j^2 \, dz \, r^d = \frac{\omega_d r^d}{d+2} \sum_j \dot{a}_j^2 \, ,$$

where $\omega_d = |B(0,1)|$ is the volume of the unit ball in $\mathbb{R}^d$. For a geodesic, this expression is constant for $t \in [0,1]$ and equals the action $\mathcal{A}_a$ in (3.6) for the ellipsoidal Euler droplet.

3.4. **Ellipsoidal Wasserstein droplets.** Let (Q, ϕ, p) be an ellipsoidal Euler droplet solution as given by Proposition 3.4, so that $\Omega_0 = E_{a(0)}$ and $\Omega_1 = E_{a(1)}$ are co-axial ellipsoids. We will call the optimal transport map T between these co-axial ellipsoids an *ellipsoidal Wasserstein droplet*. This is described and related to the Euler droplet as follows.

Given $A \in \mathbb{R}^d$, let $D_A = \mathrm{diag}(A_1, \ldots, A_d)$ denote the diagonal matrix with diagonal A. Then, given $\Omega_0 = E_{a(0)}$, $\Omega_1 = E_{a(1)}$ as above, the particle paths for the Wasserstein geodesic between the corresponding shape densities are given by linear interpolation via

$$T_t(z) = D_{A(t)} D_{A(0)}^{-1} z \, , \qquad A(t) = (1-t)a(0) + ta(1) \, . \tag{3.37}$$

Note that a point $z \in E_A$ if and only if $D_A^{-1} z$ lies in the unit ball $B(0,1)$ in $\mathbb{R}^d$. Thus the Wasserstein geodesic flow takes ellipsoids to ellipsoids:

$$T_t(\Omega_0) = E_{A(t)} \, , \quad t \in [0,1].$$

Let $a(t)$, $t \in [0,1]$, be the geodesic on the hyperboloid-like surface that corresponds to the Euler droplet that we started with. Recall that $\Omega_t = E_{a(t)}$ from Proposition 3.4. Because each component $t \mapsto a_j(t)$ is convex by Remark 3.5, it follows that for each $j = 1, \ldots, d$,

$$a_j(t) \le A_j(t), \quad t \in [0,1]. \tag{3.38}$$

Because $E_A = D_A B(0,1)$, we deduce from this the following important nesting property, which is illustrated in Figure 2 (where for visibility the ellipses at times $t = \frac{1}{2}$ and $t = 1$ are offset horizontally by $\frac{b}{2}$ and b respectively).

Proposition 3.8. *Given any corresponding elliptical Euler droplet and Wasserstein droplet that deform one ellipsoid $\Omega_0 = E_{a(0)}$ to another $\Omega_1 = E_{a(1)}$, the Euler domains remain nested inside their Wasserstein counterparts, with*

$$X(\Omega_0, t) = \Omega_t \subset T_t(\Omega_0), \quad t \in [0,1]. \tag{3.39}$$

Remark 3.9. In terms of the notation of this subsection, the straining flow X associated with the Euler droplet is given by $X(z,t) = D_{a(t)} D_{a(0)}^{-1} z$ in terms of the constant-speed geodesic $a(t)$ of Proposition 3.4. Due to (3.38), this flow satisfies, for each $j = 1, \ldots, d$ and $z \in \mathbb{R}^d$,

$$|X_j(z,t)| = \frac{a_j(t)}{a_j(0)} |z_j| \le \frac{A_j(t)}{A_j(0)} |z_j| = |T_t(z)_j|.$$

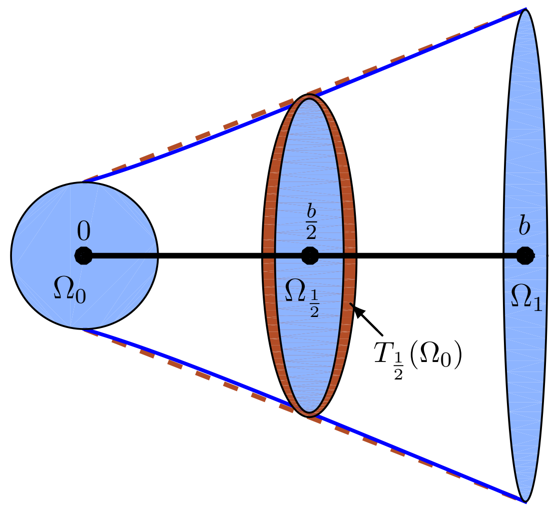

FIGURE 2. Euler droplet (light blue) deforming a circle to an ellipse, nested inside a Wasserstein droplet (dark orange). Tracks of the center and endpoints of vertical major axis are indicated for both droplets.

For the nesting property $X(\hat{\Omega}, t) \subset T_t(\hat{\Omega})$ to hold, convexity of $\hat{\Omega}$ is not sufficient in general. However, a sufficient condition is that whenever $\alpha_j \in [0, 1]$ for $j = 1, \dots, d$,

$$x = (x_1, \dots, x_d) \in \hat{\Omega} \quad \text{implies} \quad D_\alpha x = (\alpha_1 x_1, \dots, \alpha_n x_n) \in \hat{\Omega}.$$

For later use below, we describe how to bound the action for a boosted elliptical Euler droplet in terms of action for the corresponding boosted elliptical Wasserstein droplet, in the case when the source and target domains are respectively a ball and translated ellipse:

Lemma 3.10. *Given $r > 0$, $\hat{a} \in \mathbb{R}^d_+$ with $\hat{a}_1 \cdots \hat{a}_d = r^d$, and $b \in \mathbb{R}^d$, let*

$$\Omega_0 = B(0, r), \qquad \Omega_1 = E_{\hat{a}} + b\,.$$

Let $a(t)$, $t \in [0, 1]$, be the minimizing geodesic on the surface (3.25) *with*

$$a(0) = \hat{r} = (r, \dots, r), \qquad a(1) = \hat{a} = (\hat{a}_1, \dots, \hat{a}_d)\,.$$

Let (Q, ϕ, p) be the elliptical Euler droplet solution corresponding to the geodesic a, and let $\mathcal{A}_a$ denote the corresponding action. Then

$$d_W(\mathbb{1}_{\Omega_0}, \mathbb{1}_{\Omega_1})^2 \le \mathcal{A}_a \le d_W(\mathbb{1}_{\Omega_0}, \mathbb{1}_{\Omega_1})^2 + \frac{\overline{\lambda}^4}{\underline{\lambda}^2}\,\omega_d r^{d+2}\,, \tag{3.40}$$

where

$$\underline{\lambda} = \min \frac{\hat{a}_i}{r}, \qquad \overline{\lambda} = \max \frac{\hat{a}_i}{r}\,. \tag{3.41}$$

Proof. First, consider the transport cost for mapping Ω_0 to Ω_1. The (constant) velocity of particle paths starting at $x \in B(0, r)$ is

$$u(x) = (r^{-1} D_{\hat{a}} - I)x + b,$$

and the squared transport cost or action is (substituting $x = rz$)

$$d_W(\mathbb{1}_{\Omega_0}, \mathbb{1}_{\Omega_1})^2 = \int_{B(0,r)} |u(x)|^2\,dx = \sum_j \int_{B(0,r)} \left(\frac{\hat{a}_j}{r} - 1\right)^2 z_j^2 + b_j^2\,dz$$

$$= \omega_d r^d \left(|b|^2 + \frac{|\dot{A}|^2}{d+2}\right), \tag{3.42}$$

where $A(t) = (1-t)\hat{r} + t\hat{a}$ is the straight-line path from $\hat{r}$ to $\hat{a}$.

The mass density inside the transported ellipsoid $T_t(\Omega_0)$ is constant in space, given by

$$\rho(t) = \det DT_t^{-1} = \prod_i \frac{r}{A_i(t)} = \prod_i \left(1 - t + t\frac{\hat{a}_i}{r}\right)^{-1}.$$

Due to Remark 3.7, the corresponding action for the Euler droplet is bounded by that of the constant-volume path found by dilating the elliptical Wasserstein droplet: Let

$$\gamma_j(t) = \rho(t)^{1/d} A_j(t)\,.$$

Then the flow $S_t(z) = r^{-1} D_{\gamma(t)} z$ is dilational and volume-preserving (with $\prod_j \gamma_j(t) \equiv r^d$) and has zero mean velocity. The flow $z \mapsto S_t(z) + tb$ takes Ω_0 to Ω_1, as on Figure 2, with action

$$\mathcal{A}_\gamma = \int_0^1 \int_{B(0,r)} \sum_j \left(b_j + \frac{\dot{\gamma}_j z_j}{r}\right)^2 dz\,dt$$

$$= \omega_d r^d \left(|b|^2 + \frac{1}{d+2}\int_0^1 |\dot{\gamma}|^2\,dt\right). \tag{3.43}$$

Note that $\sum_j (\dot{\gamma}_j/\gamma_j)^2 \le \sum_j (\dot{A}_j/A_j)^2$, because

$$\frac{\dot{\gamma}_j}{\gamma_j} = \frac{\dot{A}_j}{A_j} + \frac{\dot{\rho}}{d\rho} = \frac{\dot{A}_j}{A_j} - \frac{1}{d}\sum_i \frac{\dot{A}_i}{A_i}.$$

Because ρ is convex we have $\rho \le 1$, hence $\gamma_j^2 \le \max A_i^2$. Thus

$$|\dot{\gamma}|^2 \le (\max A_i^2) \sum_j \frac{\dot{A}_j^2}{A_j^2} \le \left(\frac{\max A_i^2}{\min A_i^2}\right) |\dot{A}|^2 \le \left(\frac{\max \hat{a}_i^2}{\min \hat{a}_i^2}\right) |\hat{a} - \hat{r}|^2\,. \tag{3.44}$$

Plugging this back into (3.43) and using (3.42), we deduce that

$$\mathcal{A}_\gamma \le d_W(\mathbb{1}_{\Omega_0}, \mathbb{1}_{\Omega_1})^2 + \frac{\omega_d r^d}{d+2}\left(\frac{\max \hat{a}_i^2}{\min \hat{a}_i^2}\right) |\hat{a} - \hat{r}|^2\,. \tag{3.45}$$

With the notation in (3.41), $\underline{\lambda}$ and $\overline{\lambda}$ respectively are the maximum and minimum eigenvalues of DT_t, and this estimate implies

$$\mathcal{A}_a \le \mathcal{A}_\gamma \le d_W(\mathbb{1}_{\Omega_0}, \mathbb{1}_{\Omega_1})^2 + \frac{\overline{\lambda}^4}{\underline{\lambda}^2}\,\omega_d r^{d+2}\,. \tag{3.46}$$

$\square$

3.5. **Velocity and pressure estimates.** Lastly in this section we provide bounds on the velocity $v = \nabla\phi$ and pressure p for the ellipsoidal Euler droplet solutions. Note that because $1/a_j^2 \le \sum_i (1/a_i^2)$,

$$0 \le p \le \dot\beta \le \frac{1}{2}\sum_j \dot a_j^2 \le \frac{1}{2}\int_0^1 |\dot\gamma|^2\,dt$$

Using (3.44) and the notation in (3.41), it follows

$$0 \le p \le \frac{\overline{\lambda}^4}{\underline{\lambda}^2}\, r^2\,. \tag{3.47}$$

For the velocity, it suffices to note that in (3.35), $|X_j/a_j| \le 1$ hence $|\dot X|^2 \le \sum_j \dot a_j^2$. Thus the same bounds as above apply and we find

$$|\nabla\phi| \le \frac{\overline{\lambda}^4}{\underline{\lambda}^2}\, r^2. \tag{3.48}$$

Finally, for a boosted elliptical Euler droplet, with velocity boosted as in (3.20) by a constant vector $b \in \mathbb{R}^d$, the same pressure bound as above in (3.47) applies, and the same bound on velocity becomes

$$|\nabla\hat\phi - b| \le \frac{\overline{\lambda}^4}{\underline{\lambda}^2}\, r^2. \tag{3.49}$$

4. Euler sprays

Heuristically, an Euler spray is a countable disjoint superposition of solutions of the Euler droplet equations. Recall that the notation $\sqcup_n\Omega_n$ means the union of disjoint sets Ω_n.

Definition 4.1. *An **Euler spray** is a triple (Q,ϕ,p), with Q a bounded open subset of $\mathbb{R}^d \times [0,1]$ and $\phi, p : Q \to \mathbb{R}$, such that there is a sequence $\{(Q_n,\phi_n,p_n)\}_{n\in\mathbb{N}}$ of smooth solutions of the Euler droplet equations, such that $Q = \sqcup_{n=1}^\infty Q_n$ is a disjoint union of the sets Q_n, and for each $n \in \mathbb{N}$, $\phi_n = \phi|_{\Omega_n}$ and $p_n = p|_{\Omega_n}$.*

With each Euler spray that satisfies appropriate bounds we may associate a weak solution (ρ, v, p) of the Euler system (1.7)-(1.8). The following result is a simple consequence of the weak formulation in (1.9)-(1.10) together with Proposition 3.3(a) and the dominated convergence theorem.

Proposition 4.2. *Suppose (Q,ϕ,p) is an Euler spray such that $|\nabla\phi|^2$ and p are integrable on Q. Then with $\rho = \mathbb{1}_Q$ and $v = \mathbb{1}_Q\nabla\phi$ and with p extended as zero outside Q, the triple (ρ,v,p) satisfies the Euler system (1.7)-(1.8) in the sense of distributions on $\mathbb{R}^d \times [0,1]$.*

Our main goal in this section is to prove Theorem 1.2. The strategy of the proof is simple to outline: We will approximate the optimal transport map $T\colon \Omega_0 \to \Omega_1$ for the Monge-Kantorovich distance, up to a null set, by an 'ellipsoidal transport spray' built from a countable collection of ellipsoidal Wasserstein droplets. The spray maps Ω_0 to a target Ω_1^ε whose shape distance from Ω_1 is as small as desired. Then from the corresponding ellipsoidal Euler droplets nested inside, we construct the desired Euler spray that connects Ω_0 to Ω_1^ε.

4.1. **Approximating optimal transport by an ellipsoidal transport spray.** Heuristically, an ellipsoidal transport spray is a countable disjoint superposition of transport maps on ellipsoids, whose particle trajectories do not intersect.

Definition 4.3. *An **ellipsoidal transport spray** on Ω_0 is a map $S\colon \Omega_0 \to \mathbb{R}^d$, such that*

$$\Omega_0 = \bigsqcup_{n\in\mathbb{N}} \Omega_0^n$$

is a disjoint union of ellipsoids, the restriction of S to Ω_0^n is an ellipsoidal Wasserstein droplet, and the linear interpolants S_t defined by

$$S_t(z) = (1-t)z + tS(z), \qquad z \in \Omega_0,$$

remain injections for each $t \in [0,1]$.

Proposition 4.4. *Let Ω_0, Ω_1 be a pair of shapes in $\mathbb{R}^d$ of equal volume, and let $T\colon \Omega_0 \to \Omega_1$ be the optimal transport map for the Monge-Kantorovich distance with quadratic cost. For any $\varepsilon > 0$, there is an ellipsoidal transport spray $S^\varepsilon\colon \Omega_0^\varepsilon \to \mathbb{R}^d$ such that*

(i) *Ω_0^ε is a countable union of balls in the non-singular set $\Omega_0 \setminus \Sigma_0$ with $|\Omega_0 \setminus \Omega_0^\varepsilon| = 0$, and*

(ii) $\sup_{z\in\Omega_0^\varepsilon} |T(z) - S^\varepsilon(z)| < \frac{5}{8}\varepsilon \operatorname{diam} \Omega_1$.

(iii) *The L^∞ transportation distance between the uniform distributions on Ω_1^ε and Ω_1 is less than $\frac{5}{8}\varepsilon \operatorname{diam} \Omega_1$.*

The proof of this result will comprise the remainder of this subsection. The strategy is as follows. The set Ω_0^ε is chosen to be the union of a suitable Vitali covering of Ω_0 a.e. by balls. The map T is approximated on each ball by an affine map which takes the ball center x_i to $(1+\varepsilon)T(x_i)$. The dilation by $1+\varepsilon$ grants each ellipsoidal image sufficient 'personal space' to ensure the injectivity of the piecewise affine approximation.

4.1.1. *Vitali covering.* We suppose $0 < \varepsilon < 1$. The first step in the proof of the proof of Proposition 4.4 is to produce a suitable Vitali covering of Ω_0, up to a null set, by a countable disjoint union of balls. By a simple tranlation of source and target, if necessary, we may assume that $|T(x)| \le \frac{1}{2} \operatorname{diam} \Omega_1$ for all $x \in \Omega_0$.

Recall that there is a relatively closed null set $\Sigma_0 \subset \Omega_0$ such that $T = \nabla\psi$ is a smooth diffeomorphism from $\Omega_0 \setminus \Sigma_0$ to its image. Then for every $x \in \Omega_0\backslash\Sigma_0$, there exists $\overline{r}(x,\varepsilon) \in (0, \operatorname{diam} \Omega_1)$ such that whenever $0 < r < \bar{r}$, then $B(x,r) \subset \Omega_0\backslash\Sigma_0$ and both

$$\frac{\varepsilon}{4} > \frac{r\|D^3\psi\|_{B(x,r)}}{\underline{\lambda}_{B(x,r)}}, \quad \frac{\varepsilon}{8} > \left(\frac{\overline{\lambda}^2_{B(x,r)}}{\underline{\lambda}_{B(x,r)}} \frac{r}{\operatorname{diam} \Omega_1}\right)^2, \tag{4.1}$$

where $\underline{\lambda}_U$ and $\overline{\lambda}_U$ are defined by (2.11) and $\|D^3\psi\|_U$ is defined by (2.13). This follows by noting that the right-hand sides are continuous functions of r with value 0 when $r = 0$. The family of balls

$$\{B(x,r) : x \in \Omega_0\backslash\Sigma_0,\ 0 < r < \overline{r}(x,\varepsilon)\}$$

forms a Vitali cover of $\Omega_0\backslash\Sigma_0$. Therefore, by Vitali's covering theorem [19, Theorem III.12.3], there is a countable family of mutually disjoint balls $B(x_i, r_i)$, with $x_i \in \Omega_0\backslash\Sigma_0$ and $0 < r_i < \overline{r}(x_i, \varepsilon)$, such that

$$|(\Omega_0\backslash\Sigma_0)\backslash \cup_{i\in\mathbb{N}} B(x_i, r_i)| = 0 \ .$$

We let

$$\Omega_0^\varepsilon = \bigcup_{i\in\mathbb{N}} B_i\,, \qquad B_i = B(x_i, r_i). \tag{4.2}$$

For further use below, we note that $\underline{\lambda}_i \le 1 \le \overline{\lambda}_i$ for all i, where

$$\underline{\lambda}_i = \underline{\lambda}_{B(x_i,r_i)}\,, \qquad \overline{\lambda}_i = \overline{\lambda}_{B(x_i,r_i)}\,, \quad i \in \mathbb{N}. \tag{4.3}$$

We observe that from (4.1) follows

$$\|D^3\psi\|_{B_i}\, r_i < \frac{\varepsilon}{4}\underline{\lambda}_i\,. \tag{4.4}$$

4.1.2. *An approximating ellipsoidal transport spray.* We shall approximate the optimal transport map T on Ω_0^ε through linear approximation on each ball B_i, combined with a homothetic expansion of the ball centers to maintain injectivity.

For each $i \in \mathbb{N}$, we denote the linear approximation to T on B_i by

$$A^i(x) = T(x_i) + DT(x_i)(x - x_i). \tag{4.5}$$

Then we define $S^\varepsilon\colon \Omega_0^\varepsilon \to \mathbb{R}^d$ by setting, whenever $x \in B_i$,

$$\boxed{S^\varepsilon(x) = (1+\varepsilon)T(x_i) + DT(x_i)(x - x_i) = A^i(x) + \varepsilon T(x_i)\,.} \tag{4.6}$$

Because each B_i is a ball and $DT(x_i) = \operatorname{Hess}\psi(x_i)$ whose determinant is 1, the affine map A^i is an ellipsoidal Wasserstein droplet, so the same is true for the restriction of S^ε to B_i.

For every $x \in B_i$, note that we have the estimate by Taylor's theorem

$$\begin{aligned} |T(x) - S^\varepsilon(x)| &\le |T(x) - A^i(x)| + \varepsilon|T(x_i)| \\ &\le \frac{1}{2}\|D^3\psi\|_{B_i} r_i^2 + \frac{\varepsilon}{2}\operatorname{diam}\Omega_1 \\ &\le \frac{1}{8}\varepsilon\underline{\lambda}_i r_i + \frac{\varepsilon}{2}\operatorname{diam}\Omega_1 < \frac{5}{8}\varepsilon\operatorname{diam}\Omega_1\,. \end{aligned} \tag{4.7}$$

In order to show that S^ε is an ellipsoidal transport spray and complete the proof of Proposition 4.4, it remains to show that the interpolants S_t^ε defined as in Definition 4.3 are injections for each $t \in [0,1]$.

Lemma 4.5 (Injectivity of interpolants)**.** *For each $t \in [0,1]$, the interpolant $S_t^\varepsilon = (1-t)I + tS^\varepsilon$ is an injection. Its image is a union of the disjoint ellipsoids $S_t^\varepsilon(B_i)$, $i \in \mathbb{N}$, separated according to*

$$\operatorname{dist}(S_t^\varepsilon(B_i), S_t^\varepsilon(B_j)) \ge \frac{\varepsilon t}{4}(\underline{\lambda}_i r_i + \underline{\lambda}_j r_j), \qquad i \ne j. \tag{4.8}$$

Proof. Step 1. Fix $t \in [0,1]$. For each $k \in \mathbb{N}$, define

$$A_t^k = (1-t)I + tA^k, \qquad z_k = T_t(x_k), \qquad E_k = A_t^k(B_k),$$

and note $S_t^\varepsilon = A_t^k + \varepsilon t T(x_k)$ on B_k. We first identify controlled 'central' subsets C_k of the ellipsoids E_k. Note that $z = A_t^k(x)$ if and only if

$$z - z_k = DT_t(x_k)(x - x_k). \tag{4.9}$$

If $|z - z_k| < \frac{1}{2}\underline{\lambda}_k r_k$ then $|x - x_k| < \frac{1}{2} r_k$ due to (2.12). Further, if $\hat{z} = A_t^k(\hat{x}) \notin E_k$ then $|\hat{x} - x_k| \ge r_k$ and thus $|\hat{x} - x| > \frac{1}{2} r_k$ and

$$|z - \hat{z}| = |DT_t(x_k)(x - \hat{x})| > \frac{1}{2}\underline{\lambda}_k r_k\,.$$

Now let us define

$$\delta_k = \|D^2 T_t\|_{B_k} r_k^2 = t\|D^3\psi\|_{B_k} r_k^2 \ , \tag{4.10}$$

and put

$$C_k = \{z \in E_k : \operatorname{dist}(z, \partial E_k) \ge \delta_k\} \,. \tag{4.11}$$

We deduce from (4.4) that

$$\delta_k < \frac{\varepsilon t}{4}\,\underline{\lambda}_k r_k < \frac{1}{4}\underline{\lambda}_k r_k \ , \tag{4.12}$$

and we infer from the estimate on $|z-\hat{z}|$ above that

$$B\left(z_k, \frac{1}{2}\underline{\lambda}_k r_k\right) \subset C_k \ . \tag{4.13}$$

Thus the set C_k is nonempty, and it is convex since it is the intersection of a family of closed half-spaces. Note that

$$\operatorname{dist}(z, C_k) \le \delta_k \quad \text{for all } z \in E_k. \tag{4.14}$$

We claim that C_k is contained in $T_t(B_k)$. First we show that C_k does not intersect $\partial T_t(B_k)$. For by (2.14), $z \in C_k$ and $x \in \partial B_k$ imply $A_t^k(x) \in \partial E_k$ and

$$|z - T_t(x)| \ge |z - A_t^k(x)| - |A_t^k(x) - T_t(x)| \ge \delta_k - \frac{1}{2}\delta_k > 0 \ .$$

Thus $z \notin \partial T_t(B_k)$. Now, by a path-continuation argument passing from $T_t(x_k)$ to z along a ray, it follows $C_k \subset T_t(B_k)$.

Step 2. Let $i \ne j$. We estimate the overlap of the ellipsoids $E_i = A_t^i(B_i)$ and $E_j = A_t^j(B_j)$ in a suitable direction. Note that because $T_t(B_i)$ is disjoint from $T_t(B_j)$, there is a hyperplane H that separates the disjoint convex sets C_i and C_j. Let H_i be the open half-space bounded by H containing $z_i = T_t(x_i)$; then $H_j := \mathbb{R}^d \backslash (H_i \cup H)$ contains $z_j = T_t(x_j)$. Let ν be the unit normal to H pointing from H_i to H_j.

Because $C_i \subset H_i$, by (4.14) we have

$$E_i = A_t^i(B_i) \subset H_i + \delta_i \nu \,, \qquad E_j = A_t^j(B_j) \subset H_j - \delta_j \nu \,. \tag{4.15}$$

Step 3. Finally, we prove the injectivity of S_t^ε. Note that

$$S_t^\varepsilon(B_i) \subset H_i + \delta_i \nu + \varepsilon t T(x_i) \,, \tag{4.16}$$

$$\begin{aligned} S_t^\varepsilon(B_j) &\subset H_j - \delta_j \nu + \varepsilon t T(x_i) + \varepsilon t (T(x_j) - T(x_i)) \\ &= H_j - \delta_j \nu + \varepsilon t T(x_i) + \varepsilon t \nu\, \nu \cdot (T(x_j) - T(x_i)) \,. \end{aligned} \tag{4.17}$$

Let z_H be the point of intersection of the hyperplane H with the line passing through z_i and z_j. Then due to (4.13), necessarily we have

$$\frac{1}{2}\underline{\lambda}_i r_i \le \nu \cdot (z_H - z_i) \,, \qquad \frac{1}{2}\underline{\lambda}_j r_j \le \nu \cdot (z_j - z_H) \,, \tag{4.18}$$

Multiply these inequalities by εt, add them and substitute into (4.17). Using (4.12) we deduce

$$S_t^\varepsilon(B_j) \subset H_j + \delta_i \nu + \varepsilon t z_i + \nu \frac{\varepsilon t}{4}(\underline{\lambda}_i r_i + \underline{\lambda}_j r_j) \,. \tag{4.19}$$

Therefore it follows that $S_t^\varepsilon(B_i)$ and $S_t^\varepsilon(B_j)$ belong to distinct hyperplanes and are separated by the distance asserted in the Lemma. □

This completes the proof of parts (i) and (ii) of Proposition 4.4. For part (iii), we note that the set $\Omega_0^\varepsilon = (S^\varepsilon)^{-1}(\Omega_1^\varepsilon)$ has full measure in $\Omega_0 \setminus \Sigma_0$, and T is a smooth diffeomorphism from this set to $\Omega_1 \setminus \Sigma_1$ so maps null sets to null sets. It follows $T \circ (S^\varepsilon)^{-1}$ maps Ω_0^ε to a set of full measure in Ω_1, satisfies

$$\sup_{x \in \Omega_1^\varepsilon} |T \circ (S^\varepsilon)^{-1}(x) - x| < \frac{5}{8}\varepsilon \operatorname{diam} \Omega_1,$$

and pushes forward uniform measure to uniform measure. The result claimed in part (iii) follows.

4.2. **Action estimate for Euler spray.** Each of the ellipsoidal Wasserstein droplets that make up the ellipsoidal transport spray S^ε is associated with a boosted ellipsoidal Euler droplet nested inside, due to the nesting property in Proposition 3.8. The disjoint superposition of these Euler droplets make up an Euler spray that deforms Ω_0^ε to the same set Ω_1^ε.

In order to complete the proof of Theorem 1.2, it remains to bound the action of this Euler spray in terms of the Wasserstein distance between the uniform measures on Ω_0 and Ω_1. Toward this goal, we first note that because the maps T and S^ε are volume-preserving, due to the estimate in part (ii) of Proposition 4.4 we have

$$d_W(T(B_i), S^\varepsilon(B_i))^2 \le \left(\frac{5\varepsilon}{8} K_1\right)^2 |B_i|\,, \qquad K_1 = \operatorname{diam} \Omega_1.$$

(One obtains this by bounding the transport cost of straight-line motion from $T(z)$ to $S^\varepsilon(z)$ using the Lagrangian form of the action in (3.6).) Now by the triangle inequality,

$$\begin{aligned} d_W(B_i, S^\varepsilon(B_i))^2 &\le \left(d_W(B_i, T(B_i)) + \frac{5}{8}\varepsilon K_1 |B_i|^{1/2}\right)^2 \\ &\le d_W(B_i, T(B_i))^2(1+\varepsilon) + 2\varepsilon \left(\frac{5}{8} K_1\right)^2 |B_i| \end{aligned} \tag{4.20}$$

Recall that by inequality (3.40) of Lemma 3.10, the action of the ith ellipsoidal Euler droplet, denoted by $\mathcal{A}_i$, satisfies

$$\begin{aligned} \mathcal{A}_i &\le d_W(B_i, S^\varepsilon(B_i))^2 + \frac{\overline{\lambda}_i^4}{\underline{\lambda}_i^2} r_i^2 |B_i| \\ &\le d_W(B_i, T(B_i))^2(1+\varepsilon) + \varepsilon K_1^2 |B_i|\,, \end{aligned} \tag{4.21}$$

where we make use of the second constraint in (4.1).

By summing over all i, we obtain the required bound,

$$\mathcal{A}^\varepsilon = \sum_i \mathcal{A}_i \le d_W(\mathbb{1}_{\Omega_0}, \mathbb{1}_{\Omega_1})^2 + K\varepsilon$$

where

$$K = d_W(\mathbb{1}_{\Omega_0}, \mathbb{1}_{\Omega_1})^2 + |\Omega_0| (\operatorname{diam} \Omega_1)^2.$$

This concludes the proof of Theorem 1.2.

4.3. **Displacement interpolants as weak limits.** Next we supply the proof of Theorem 1.3. First we describe the bounds on pressure and velocity that come from the construction of the Euler sprays constructed above, for any given $\varepsilon \in (0,1)$.

Lemma 4.6. *Let $(Q^\varepsilon, \phi^\varepsilon, p^\varepsilon)$, $0<\varepsilon<1$, denote the Euler sprays constructed in the proof of Theorem 1.2, and let $X^\varepsilon\colon \Omega_0^\varepsilon \times [0,1] \to \mathbb{R}^d$ denote the associated flow maps, which satisfy*

$$\dot X^\varepsilon(z,t) = \nabla\phi^\varepsilon(X^\varepsilon(z,t),t), \quad (z,t)\in \Omega_0^\varepsilon\times[0,1],$$

with $X^\varepsilon(z,0)=z$. Then for some $\hat K>0$ independent of ε, we have

$$0 \le p^\varepsilon(x,t) \le \hat K \varepsilon \tag{4.22}$$

for all $(x,t)\in Q^\varepsilon$, and

$$|X^\varepsilon(z,t) - T_t(z)| + |\dot X^\varepsilon(z,t) - \dot T_t(z)| \le \hat K\sqrt{\varepsilon} \tag{4.23}$$

for all $(z,t) \in \Omega_0^\varepsilon \times [0,1]$, where $(z,t)\mapsto T_t(z)$ is the flow map from (2.1) for the Wasserstein geodesic.

Proof. By the pressure bound for individual droplets in (3.47) together with the second condition in (4.1), we have the pointwise bound

$$0 \le p^\varepsilon \le K_0\varepsilon\,, \qquad K_0 = \frac{1}{8}K_1^2. \tag{4.24}$$

Next consider the velocity. The boosted elliptical Euler droplet that transports B_i to $S^\varepsilon(B_i)$ is translated by x_i, and boosted by the vector

$$b_i := (1+\varepsilon)T(x_i) - x_i = \dot T_t(x_i) + \varepsilon T(x_i)\,. \tag{4.25}$$

In this "ith droplet," the velocity satisfies, by the estimate (3.49),

$$|\nabla\phi^\varepsilon - b_i| = |v^\varepsilon - b_i| \le K_0\varepsilon\,. \tag{4.26}$$

Now the particle velocity for the Euler spray compares to that of the Wasserstein geodesic according to

$$\begin{aligned} |\dot X^\varepsilon(z,t) - \dot T_t(z)| &\le |\dot X^\varepsilon - b_i| + |b_i - \dot T_t(z)| \\ &\le K_0\varepsilon + \varepsilon|T(x_i)| + r_i \max_j |\lambda_j(z)-1| \\ &\le K_0\varepsilon + K_1\varepsilon + \sqrt{K_0\varepsilon} \le K_2\sqrt{\varepsilon}\,. \end{aligned} \tag{4.27}$$

(Here $\lambda_j(z)$ denote the eigenvalues of $DT(z)=\nabla\psi(z)$, and we use the fact that $|\lambda_j(z)-1| \le \overline{\lambda}_i$ together with (4.1).) Upon integration in time we obtain both bounds in (4.23). □

Proof of Theorem 1.3. Now, let (ρ, v) be the density and velocity of the particle paths for the Wasserstein geodesic, from (2.5) and (2.3). To prove $\rho^\varepsilon \overset{\star}{\rightharpoonup} \rho$ weak-$\star$ in L^∞, it suffices to show that as $\varepsilon\to 0$,

$$\int_0^1 \int_{\mathbb{R}^d} (\rho^\varepsilon - \rho) q \,dx\,dt \to 0\,, \tag{4.28}$$

for every smooth test functions $q \in C_c^\infty(\mathbb{R}^d\times[0,1],\mathbb{R})$. Changing to Lagrangian variables using X^ε for the term with $\rho^\varepsilon = \mathbb{1}_{Q^\varepsilon}$ and T_t for the term with ρ, the left-hand side becomes

$$\int_0^1 \int_{\Omega_0} (q(X^\varepsilon(z,t),t) - q(T_t(z),t))\,dz\,dt\,. \tag{4.29}$$

Evidently this does approach zero as $\varepsilon\to 0$, due to (4.23).

Next, we claim $\rho^\varepsilon v^\varepsilon \overset{\star}{\rightharpoonup} \rho v$ weak-$\star$ in L^∞. Because these quantities are uniformly bounded, it suffices to show that as $\varepsilon \to 0$,

$$\int_0^1 \int_{\mathbb{R}^d} (\rho^\varepsilon v^\varepsilon - \rho v) \cdot \tilde{v}\, dx\, dt \to 0 \tag{4.30}$$

for each $\tilde{v} \in C_c^\infty(\mathbb{R}^d \times [0,1], \mathbb{R}^d)$. Changing variables in the same way, the left-hand side becomes

$$\int_0^1 \int_{\Omega_0} \Big(\dot{X}^\varepsilon(z,t) \cdot \tilde{v}(X^\varepsilon(z,t),t) - \dot{T}_t(z) \cdot \tilde{v}(T_t(z),t) \Big)\, dz\, dt\,. \tag{4.31}$$

But because $\tilde{v}$ is smooth and due to the bounds in (4.23), this also tends to zero as $\varepsilon \to 0$.

It remains to prove $\rho^\varepsilon v^\varepsilon \otimes v^\varepsilon \overset{\star}{\rightharpoonup} \rho v \otimes v$ weak-$\star$ in L^∞. Considering the terms componentwise, the proof is extremely similar to the previous steps. This finishes the proof of Theorem 1.3. □

Remark 4.7. In [24] the authors introduced a way to measure differences between functions defined with respect to different measures, which extends the notion of L^p convergence. The associated metric on the space of ordered pairs (μ, g) where μ is a probability measure and $g \in L^p(\mu)$ is the TL^p metric: For $1 \le p < \infty$,

$$d_{TL^p}((\mu_0, g_0), (\mu_1, g_1)) = \inf_{\pi \in \Pi(\mu_0,\mu_1)} \iint |x-y|^p + |g_0(x) - g_1(y)|^p d\pi(x,y)$$

and

$$d_{TL^\infty}((\mu_0, g_0), (\mu_1, g_1)) = \inf_{\pi \in \Pi(\mu_0,\mu_1)} \operatorname*{ess\,sup}_\pi (|x-y| + |g_0(x) - g_0(y)|)$$

where $\Pi(\mu_0, \mu_1)$ is the set of transportation plans between μ_0 and μ_1.

From Lemma 4.6 follows that for $1 \le p \le \infty$,

$$(\rho_\varepsilon, v_\varepsilon) \xrightarrow{TL^p} (\rho, v) \quad \text{and} \quad (\rho_\varepsilon, v_\varepsilon \otimes v_\varepsilon) \xrightarrow{TL^p} (\rho, v \otimes v) \tag{4.32}$$

as $\varepsilon \to 0$, uniformly for $t \in [0,1]$. Namely, using the transport plan given in terms of the map T_t by

$$\pi = (X^\varepsilon(\,\cdot\,, t) \times T_t)_\sharp \rho_0\,,$$

the estimate (4.23) implies that for π-a.e. (x,y),

$$|x-y| + |v_\varepsilon(x,t) - v(y,t)| \le \hat{K}\sqrt{\varepsilon}$$

for all $t \in [0,1]$. This implies a somewhat stronger convergence of approximate velocities to v than was used in the proof of Theorem 1.3 above.

5. Smooth sprays and shape distance between open sets

Our main goal in this section is to prove Theorem 1.1. We first treat the case when both Ω_0 and Ω_1 are bounded open sets (in subsections 5.1 and 5.2). This will be done by constructing a collection of paths of shape densities connecting the source Ω_0 to the exact target Ω_1, which approximate Wasserstein geodesics in some sense. The key idea is to decompose Ω_0 as a countable collection of 'microdroplets,' and transport them separately by smooth incompressible flows to their targets under the Brenier map T. That this can be done without overlaps is due to the fact that the displacement interpolants T_t *expand* volume.

Let us say that a path of shape densities $\rho = (\rho_t)_{t\in[0,1]}$ is *smooth* if the support of ρ_t is a set with smooth boundary for each t and ρ is transported by a velocity field v that is smooth on the support of ρ. Heuristically, a smooth spray is a countable disjoint superposition of smooth paths of shape densities.

Definition 5.1. *A* smooth spray *is a path of shape densities $\rho = (\rho_t)_{t\in[0,1]}$ which is a disjoint superposition of a countable collection of smooth paths of shape densities $\rho^n = (\rho^n_t)_{t\in[0,1]}$, satisfying*

$$\rho = \sum_n \rho^n \quad a.e.$$

We note that, by superposition, if ρ^n is transported by velocity field u^n for each n, then ρ is transported by the velocity field $u = \sum_n u^n$ if ρu is integrable.

Theorem 5.2. *Let Ω_0, Ω_1 be open shapes in $\mathbb{R}^d$ of equal volume. For any $\varepsilon > 0$, there exists a smooth spray $\rho = (\rho_t)$ connecting $\rho_0 = \mathbb{1}_{\Omega_0}$ to $\rho_1 = \mathbb{1}_{\Omega_1}$, which is transported by a velocity field u satisfying*

$$d_w(\mathbb{1}_{\Omega_0}, \mathbb{1}_{\Omega_1})^2 \le \int_0^1 \int_{\mathbb{R}^d} \rho|u|^2\,dx\,dt \le d_w(\mathbb{1}_{\Omega_0}, \mathbb{1}_{\Omega_1})^2 + \varepsilon. \tag{5.1}$$

The conclusion of Theorem 1.1 in case Ω_0 and Ω_1 are open follows as a direct consequence of Theorem 5.2. Note that the path of measures $t \mapsto \sigma_t = \rho_t\,dx$ is necessarily weak-$\star$ continuous, as a consequence of [2, Theorem 8.3.1].

5.1. **Incompressible deformation of balls.** Let $T = \nabla\psi$ be the optimal transport map between Ω_0 and Ω_1 as before. Our first goal is to produce, for any given open ball $O_0 = B(x_0, r)$ with compact closure in the regular set $\Omega_0\backslash\Sigma_0$ of T, an incompressible velocity field u that deforms this ball exactly onto its image $T(O_0)$. For small enough r, the cost will be close to the Wasserstein optimal cost, and the incompressible flow will keep the ball *inside* its image under the displacement interpolant.

Let $v_0(x) = T(x) - x$ be the (constant) velocity along the Wasserstein particle path starting at x, given by the displacement interpolant

$$T_t(x) = x + tv_0(x)\,.$$

As a preliminary step, we dilate the image $T_t(O_0)$ about the point $T_t(x_0)$ to maintain constant volume. After that we adjust the velocity field to obtain an incompressible flow. Define a 'shrunken' flow map for $x \in O_0$ by

$$S_t(x) = a(t)T_t(x) + (1 - a(t))T_t(x_0)\,, \quad a(t) = \left(\frac{|T_t(O_0)|}{|O_0|}\right)^{-1/d}. \tag{5.2}$$

The image $O_t = S_t(O_0)$ has constant volume $|O_t| = |O_0|$. Note that $a(0) = a(1) = 1$, and $0 < a(t) \le 1$ for all $t \in [0,1]$, because

$$|T_t(O_0)| = \int_{T_t(O_0)} dz = \int_{O_0} \frac{1}{\rho(T_t(x),t)}\,dx \ge |O_0|.$$

The map $(x,t) \mapsto S_t(x)$ is smooth in the space-time domain $\bar{Q}$ where

$$Q = \bigcup_{t\in[0,1]} O_t \times \{t\} \quad \subset \mathbb{R}^d \times [0,1].$$

Figure 3 illustrates the volume preserving path from a single ball O_0 to its image $T(O_0) = O_1$, with a snapshot of $T_t(O_0)$ and $O_t = S_t(O_0)$ at $t = \frac{1}{2}$.

The Eulerian velocity field associated with S_t is given by

$$w(S_t(x), t) = \frac{d}{dt} S_t(x)\,. \tag{5.3}$$

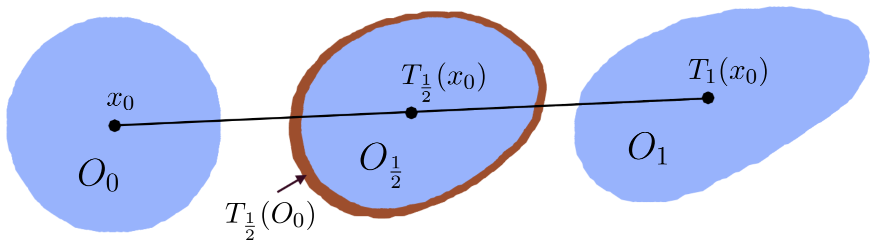

FIGURE 3. Illustration of the volume preserving flow from a ball O_0, centered at x_0, to its image under the transport map, $T(O_0) = O_1$. Slices at $t = 0, \frac{1}{2}$, and 1 are shown in light blue. The domain at time t, O_t is obtained by the dilating the displacement interpolant $T_t(O_0)$ (shown in dark orange) about $T_t(x_0)$. (Dilation is enhanced to improve visibility.)

This velocity field need not be divergence-free. We find a divergence-free velocity field whose flow map induces the same family of images O_t by setting

$$u = \nabla \phi$$

where

$$\Delta \phi = 0 \quad \text{in } O_t\,, \qquad \nabla \phi \cdot \nu = w \cdot \nu \quad \text{on } \partial O_t\,. \tag{5.4}$$

Note that compatibiility holds: $\int_{\partial O_t} w \cdot \nu = (d/dt)|O_t| = 0$, so a zero-mean solution ϕ to this boundary-value problem exists, and provides a smooth function on $\bar{Q}$.

For use below, we note that w is a spatial gradient: Writing $z_0(t) = (1 - a(t))T_t(x_0)$, we have

$$w(S_t(x), t) = a' T_t(x) + a v(T_t(x), t) + z_0'(t)$$

whence it follows that with $z = S_t(x)$,

$$w(z, t) = \frac{a'}{a}(z - z_0) + a\, v\left(\frac{z - z_0}{a}, t\right) + z_0' \tag{5.5}$$

Due to 2.9, evidently this is spatially a gradient.

Recalling that $O_o = B(x_0, r)$, for simplicity we write

$$\underline{\lambda}_r = \underline{\lambda}_{O_0}\,, \quad \overline{\lambda}_r = \overline{\lambda}_{O_0}\,, \tag{5.6}$$

where $\underline{\lambda}_U$ and $\overline{\lambda}_U$ are defined in (2.11).

Proposition 5.3. *Let $x_0 \in \Omega_0 \backslash \Sigma_0$ and $r \in (0,1)$ such that $O_0 := B(x_0, r)$ has compact closure in $\Omega_0 \backslash \Sigma_0$. Then, with $O_t = S_t(O_0)$ and u determined as above, the path in the space of measures given by*

$$t \mapsto \sigma_t\,, \quad d\sigma_t = \mathbb{1}_{O_t} dx\,,$$

is absolutely continuous, the vector field u is smooth on $\bar{Q}$, and we have

$$\partial_t \sigma + \nabla \cdot (\sigma u) = 0, \quad \nabla \cdot u = 0\,,$$

and for all $t \in [0,1]$,

$$\int_{O_t} |u(y,t) - v_0(x_0)|^2 dy \le C(\underline{\lambda}_r, \overline{\lambda}_r)\, r^{d+2}\,, \tag{5.7}$$

and

$$\int_{O_t} |u(y,t)|^2 dy \le (1+r)\int_{O_0} |v_0(x)|^2 dx + C(\underline{\lambda}_r, \overline{\lambda}_r)\, r^{d+1}. \tag{5.8}$$

Furthermore, $O_t \subseteq T_t(O_0)$ for all $t \in [0,1]$, provided $r > 0$ is so small that

$$r\|D^3\psi\|_{B(x_0,r)} < \underline{\lambda}_r. \tag{5.9}$$

The proof of this proposition is broken into several lemmas, and will be concluded at the end of this subsection. We begin with a few basic notions and estimates.

Below, given $O_0 = B(x_0, r) \subset \Omega_0 \backslash \Sigma_0$ we will write

$$U_t = T_t(O_0).$$

Recall $T_t = \nabla\psi_t$ in terms of the corresponding transportation potential

$$\psi_t(x) = (1-t)\frac{|x|^2}{2} + t\psi(x)\,.$$

Note that the eigenvalues of Hess ψ_t on O_0 are bounded by $\underline{\lambda}_r$ and $\overline{\lambda}_r$ respectively from below and above. For all $y \in O_0$, $t \in [0,1]$ and $z \in \mathbb{R}^d$,

$$\underline{\lambda}_r|z| \le |DT_t(y)z| \le \overline{\lambda}_r|z|\,. \tag{5.10}$$

Lemma 5.4. *The divergence of the velocity w in (5.3) is uniformly bounded, with*

$$\sup_{t\in[0,1]}\sup_{O_t}|\nabla\cdot w| \le C = C(\underline{\lambda}_r, \overline{\lambda}_r). \tag{5.11}$$

Moreover

$$|w(S_t(x),t) - v_0(x_0)| \le C\overline{\lambda}_r\, r\,. \tag{5.12}$$

Proof. After recalling that

$$v(T_t(x),t) = \frac{d}{dt}T_t(x) = v_0(x),$$

we can write

$$w(S_t(x),t) = v_0(x_0) + a(t)(v_0(x) - v_0(x_0)) + a'(t)(T_t(x) - T_t(x_0))\,.$$

Noting $T_t(x) - T_t(x_0) = (S_t(x) - S_t(x_0))/a(t)$ and changing to Eulerian variables $z = S_t(x)$, because $\nabla\cdot z = d$ we find that the Eulerian divergence

$$\nabla\cdot w = \nabla\cdot v(T_t(x),t) + \frac{a'(t)d}{a(t)} = -\frac{d}{dt}\log(\rho|U_t|) \tag{5.13}$$

due to Eq. (2.7) and the definition of $a(t)$. From (2.7) we have the bound

$$\left|\frac{1}{\rho}\frac{d\rho}{dt}\right| \le d\max\left(\frac{1-\underline{\lambda}_r}{\underline{\lambda}_r}, \overline{\lambda}_r - 1\right) =: C_1\,.$$

Because

$$|U_t| = \int_{O_0}\frac{1}{\rho(T_t(x),t)}\,dx\,, \tag{5.14}$$

we have

$$\left|\frac{a'(t)d}{a(t)}\right| = \frac{1}{|U_t|}\left|\frac{d}{dt}|U_t|\right| \le \frac{1}{|U_t|}\int_{O_0}\frac{1}{\rho}\left|\frac{1}{\rho}\frac{d\rho}{dt}\right|\,dx \le C_1\,. \tag{5.15}$$

Hence $|\nabla\cdot w| \le 2C_1$.

Clearly

$$|v_0(x) - v_0(x_0)| \le \overline{\lambda}_r r\,. \tag{5.16}$$

Because $|a'd/a| \le C_1$ and $a \le 1$, by (5.10) we infer (5.12). □

Lemma 5.5. *Let $u = \nabla\phi$ be determined from w by* (5.4). *Then*

$$\int_{O_t} |u-w|^2 dx \le C(\underline{\lambda}_r, \overline{\lambda}_r) r^{d+2}\,, \tag{5.17}$$

and

$$\|u\|_{O_t} := \left(\int_{O_t} |u|^2 dx\right)^{1/2} \le \|w\|_{O_t} + C(\underline{\lambda}_r, \overline{\lambda}_r) r^{(d+2)/2}. \tag{5.18}$$

Proof. Recall from (5.5) that for fixed t, w is a spatial gradient, which we write as $w = \nabla p$ for simplicity. Let $q = \phi - p$ (with ϕ from (5.4)) have zero mean. Then $q \in H^1(O_t)$ satisfies

$$\Delta q = -\nabla \cdot w \quad \text{in } O_t, \qquad \nabla q \cdot \nu = 0 \quad \text{on } \partial O_t. \tag{5.19}$$

We now estimate $\nabla q = u - w$ using (5.11) and the Poincaré inequality (5.21) in the Lemma below, finding that

$$\begin{aligned}
\int_{O_t} |\nabla q|^2 dx &= -\int_{O_t} q\Delta q\, dx = \int_{O_t} q(\nabla\cdot w)\, dx \\
&\le \left(\int_{O_t} q^2 dx\right)^{1/2} \left(\int_{O_t} (\nabla\cdot w)^2\right)^{1/2} \\
&\le C(\underline{\lambda}_r, \overline{\lambda}_r) r \left(\int_{O_t} |\nabla q|^2 dx\right)^{1/2} r^{d/2}.
\end{aligned}$$

Therefore

$$\int_{O_t} |\nabla q|^2 dx \le C(\underline{\lambda}_r, \overline{\lambda}_r)\, r^{d+2}. \tag{5.20}$$

This yields the bound (5.17), and (5.18) follows directly. □

Lemma 5.6. *There exists $C = C(\overline{\lambda}_r)$ such that for all $t \in [0,1]$ and all $g \in H^1(O_t)$ with mean $\bar{g}$,*

$$\int_{O_t} |g - \bar{g}|^2 dx \le Cr^2 \int_{O_t} |\nabla g|^2 dx. \tag{5.21}$$

Proof. Let $g \in H^1(O_t) \cap C^\infty(O_t)$ and let $f = g \circ T_t$. Then for $x \in O_0 = B(x_0, r)$,

$$\nabla f(x) = DT_t(x) \nabla g(T_t(x)),$$

and thus by (5.10) for $y = T_t(x)$

$$|\nabla g(y)| = |DT_t(x)^{-1} \nabla f(x)| \ge (\overline{\lambda}_r)^{-1} |\nabla f(x)|.$$

Let us also recall from (2.5) that $1 \le \det DT_t(x) \le \overline{\lambda}_r^d$. Therefore by the usual Poincaré inequality on $B(x_0, r)$,

$$\begin{aligned}
\int_{O_t} |\nabla g(y)|^2 dy &= \int_{B(x_0,r)} |DT_t(x)^{-1} \nabla f(x)|^2 \det DT_t(x)\, dx \\
&\ge (\overline{\lambda}_r)^{-2} \int_{B(x_0,r)} |\nabla f(x)|^2 dx \\
&\ge r^{-2} (\overline{\lambda}_r)^{-2} \int_{B(x_0,r)} |f(x) - \bar{f}|^2 dx \\
&\ge r^{-2} (\overline{\lambda}_r)^{-d-2} \int_{O_t} |g(y) - \bar{f}|^2 dy \\
&\ge r^{-2} (\overline{\lambda}_r)^{-d-2} \int_{O_t} |g(y) - \bar{g}|^2 dy\,.
\end{aligned}$$

□

Proof of the Proposition. Note $\nabla \cdot u = 0$ and $u \cdot \nu = w \cdot \nu$ on ∂O_t. Thus $\partial_t \sigma + \nabla \cdot (\sigma u) = 0$.

The inequality (5.7) follows by the triangle inequality from (5.17) and (5.12). Using (5.12) and (5.16) and the fact that $|O_t| = |O_0| = \omega_d r^d$, we find

$$\|w\|_{O_t} \le \|v_0(x_0)\|_{O_0} + Cr|O_0|^{1/2} \le \|v_0\|_{O_0} + Cr^{1+d/2}\,.$$

Using this in (5.18), because $C\alpha r^{1+d/2} \le \alpha^2 r + 4C^2 r^{d+1}$, the desired inequality (5.8) follows after squaring.

We now prove that for each x_0 fixed, if $r > 0$ is small enough, then

$$O_t = S_t(O_0) \subseteq U_t = T_t(O_0)$$

for all $t \in [0, 1]$. Because $a(t) \in (0, 1]$, It suffices to show that U_t is star-shaped about $T_t(x_0)$. Because T_t is a diffeomorphism on a neighborhood of $\bar{O}_0$, for this it suffices to show that for each boundary point $x \in \partial O_0$,

$$\alpha(x) := (T_t(x) - T_t(x_0)) \cdot \nu(x) > 0\,,$$

where $\nu(x)$ is the outside unit normal to ∂U_t at $T_t(x)$. (For, when moving away from $T_t(x_0)$ along any ray, one must then leave U_t at any boundary point encountered, and cannot re-enter.) Note that $\alpha(x) > 0$ at points where $|T_t(x) - T_t(x_0)|$ is maximized, because $\nu(x)$ is parallel to $T_t(x) - T_t(x_0)$. Thus, to show $\alpha(x) > 0$ everywhere it suffices to show that $\alpha(x) \ne 0$ on ∂U_t.

Suppose instead $\alpha(x) = 0$ where $|x - x_0| = r$. Then the vector $T_t(x) - T_t(x_0)$ is tangent to ∂U_t, and there must exist a tangent vector z to ∂O_0 at x such that

$$DT_t(x)z = T_t(x) - T_t(x_0) = (DT_t(x) + E(x))(x - x_0),$$

where $E(x)$ $(= \int_0^1 (DT_t(x_\tau) - DT_t(x))\, d\tau$ with $x_\tau = x_0 + \tau(x - x_0))$ satisfies the bound

$$|E(x)| \le M_r |x - x_0|, \quad M_r = \|D^3 \psi\|_{B(x_0,r)}\,.$$

Now $z \perp x - x_0$, so $r = |x - x_0| \le |x - x_0 - z|$, and we infer by (5.10) that

$$\underline{\lambda}_r r \le |DT_t(x)(x - x_0 - z)| \le M_r r^2$$

hence $M_r r / \underline{\lambda}_r \ge 1$. But this contradicts the stated condition. □

5.2. **Proof of Theorem 5.2.**

Proof. Let $\varepsilon > 0$ and $D = d_w(\mathbb{1}_{\Omega_0}, \mathbb{1}_{\Omega_1})^2$. We can assume that $D > 0$. For any $x_0 \in \Omega_0 \backslash \Sigma_0$ let $\bar{r}(x_0) > 0$ be small enough for condition (5.9) of the Proposition to hold. Let $0 < r_0(x, \varepsilon) < \bar{r}(x)$ be such that

$$r_0(x,\varepsilon) \leq \min\left\{\frac{1}{D+1}, \frac{\omega_d}{|\Omega_0|\, C(\underline{\lambda}_{\bar{r}}, \overline{\lambda}_{\bar{r}})}\right\} \frac{\varepsilon}{2}\,. \tag{5.22}$$

By the Vitali covering theorem, there is a countable family of disjoint balls $B(x_i, r_i)$ contained in $\Omega_0 \backslash \Sigma_0$ such that $r_i < r_0(x_i, \varepsilon)$ and

$$|(\Omega_0 \backslash \Sigma_0) \backslash \cup_{i=1}^{\infty} B(x_i, r_i)| = 0\,.$$

Let $O_{x_i,t}$ be the sets constructed in Proposition 5.3 corresponding to $x_0 = x_i$, $r = r_i$, and let u_{x_i} be the corresponding divergence-free velocity field described in Proposition 5.3. Then let

$$O_t = \bigcup_{i \in \mathbb{N}} O_{x_i,t}\,, \quad \rho_t = \mathbb{1}_{O_t}\,, \quad u = \sum_{i=1}^{\infty} u_{x_i} \mathbb{1}_{O_{x_i,t}}\,.$$

Because $\rho u = u\, dx$, the continuity equation for ρ holds by linearity.

To prove (5.1) we note that the first inequality follows form the characterization of Wasserstein distance by Benamou and Brenier [4]. To obtain the second inequality we estimate using Proposition 5.3

$$\begin{aligned} \int_0^1 \int_{O_t} |u(y,t)|^2 dy\, dt &= \sum_{i=1}^{\infty} \int_0^1 \int_{O_{x_i,t}} |u(y,t)|^2 dy\, dt \\ &\leq (1 + \max_i r_i) \int_{\Omega_0} |v_0(x)|^2 dx + \sum_{i=1}^{\infty} C(\underline{\lambda}_{\bar{r}(x_i)}, \overline{\lambda}_{\bar{r}(x_i)})\, r_i^{d+1} \\ &\leq D + \frac{\varepsilon}{2} + \frac{\varepsilon}{2}. \end{aligned} \tag{5.23}$$

It follows that the path $t \mapsto \rho_t$ is a smooth spray transported by the velocity field u satisfying (5.1), as claimed. $\square$

5.3. **Shape distance for arbitrary measurable shapes.** Our objective in this subsection is to finish the proof of Theorem 1.1 by treating the case when Ω_0 and Ω_1 are arbitrary bounded measurable sets with positive, equal volume. It will suffice to treat the case when one of sets, say Ω_0, is open.

Lemma 5.7. *Let Ω_1 be a bounded measurable set. There is a sequence of uniformly bounded open sets $\hat{\Omega}_k$, $k = 1, 2, \ldots$, such that the volume $|\hat{\Omega}_k| = |\Omega_1|$ for all k and $d_W(\mathbb{1}_{\hat{\Omega}_k}, \mathbb{1}_{\Omega_1}) \to 0$ as $k \to \infty$.*

Proof of Theorem 1.1. We postpone the proof of this lemma and first finish the proof of Theorem 1.1. Let $\varepsilon \in (0, \frac{1}{2})$ and put $\rho_0 = \mathbb{1}_{\Omega_0}$, $\rho_1 = \mathbb{1}_{\Omega_1}$. Using Lemma 5.7 and passing to a subsequence, writing $\hat{\rho}_k = \mathbb{1}_{\hat{\Omega}_k}$ we may assume that

$$d_W(\hat{\rho}_k, \rho_1) < \varepsilon 2^{-k-1}\,, \quad k = 1, 2, \ldots,$$

so that $d_W(\hat{\rho}_k, \hat{\rho}_{k+1}) < \varepsilon 2^{-k}$ for $k = 1, 2, \ldots$.

Concatenation of paths. We will construct a path of shape densities $\rho = (\rho_t)_{t\in[0,1]}$ connecting ρ_0 to ρ_1 by concatenating smooth sprays that connect $\hat{\rho}_k$ to $\hat{\rho}_{k+1}$ for $k = 0, 1, \ldots$.

For $k = 0, 1, \ldots$, let $\rho^k = (\rho^k_t)_{t\in[0,1]}$ be a smooth spray as given by Theorem 5.2 that connects $\hat\rho_k$ to $\hat\rho_{k+1}$, with v^k the corresponding velocity field, such that

$$\int_0^1 \int_{\mathbb{R}^d} \rho^0_t |v^0|^2 \, dx \, dt \le d_W(\rho_0, \hat\rho_1)^2 + \varepsilon, \tag{5.24}$$

and for $k = 1, 2, \ldots$,

$$\int_0^1 \int_{\mathbb{R}^d} \rho^k_t |v^k|^2 \, dx \, dt \le d_W(\hat\rho_k, \hat\rho_{k+1})^2 + (\varepsilon 2^{-k-1})^2 < (\varepsilon 2^{-k})^2. \tag{5.25}$$

Now, for $k = 1, 2 \ldots$, let $\tau_k = \varepsilon 2^{-k}$, so $\sum_{k\ge1} \tau_k = \varepsilon$. Put $\tau_0 = 1 - \varepsilon$, and

$$t_0 = 0, \quad t_{k+1} = t_k + \tau_k \quad \text{for } k \ge 0.$$

Define ρ_t and $v(\cdot, t)$ for $t \in [0, 1]$ by setting ρ_t for $t \in [t_k, t_{k+1})$ as

$$\rho_t = \rho^k_s, \quad v(\cdot, t) = \tau_k^{-1} v^k(\cdot, s) \quad \text{for } t = t_k + s\tau_k,\ s \in [0, 1).$$

Evidently we have $t \mapsto \rho_t$ continuous with respect to Wasserstein distance on $[0, 1]$, which implies weak-$\star$ continuity [45, Thm. 7.12]. Moreover, ρ_t is transported by the velocity field v, satisfying the continuity equation, and

$$\begin{aligned}
\int_0^1 \int_{\mathbb{R}^d} \rho_t |v|^2 \, dx \, dt &= \sum_{k=0}^{\infty} \int_{t_k}^{t_{k+1}} \int_{\mathbb{R}^d} \rho_t |v|^2 \, dx \, dt \\
&= \sum_{k=0}^{\infty} \int_0^1 \int_{\mathbb{R}^d} \rho^k_s |v^k|^2 \, dx \, ds \, \tau_k^{-1} \\
&\le (d_W(\rho_0, \hat\rho_1)^2 + \varepsilon)\tau_0^{-1} + \sum_{k=1}^{\infty} (\varepsilon 2^{-k})^2 \tau_k^{-1} \\
&\le d_W(\rho_0, \rho_1)^2 + K\varepsilon. \qquad (5.26)
\end{aligned}$$

This completes the proof of Theorem 1.1. □

Proof of Lemma 5.7. We recall that weak-$\star$ convergence of probability measures supported in a fixed compact set is equivalent to convergence in Wasserstein distance. Given Ω_1 bounded and (Lebesgue) measurable, due to outer regularity there is a sequence of uniformly bounded open sets $O_k \supset \Omega_1$ such that $|O_k \setminus \Omega_1| \to 0$. We define $\hat\Omega_k$ by dilation:

$$\hat\Omega_k = O_k c_k, \qquad c_k = \frac{|\Omega_1|^{1/d}}{|O_k|^{1/d}}.$$

Then $|\hat\Omega_k| = 1$ for all k, $c_k \to 1$, and the characteristic functions $\hat\rho_k = \mathbb{1}_{\hat\Omega_k}$ converge weak-$\star$ to $\rho_1 = \mathbb{1}_{\Omega_1}$, because for any continuous test function f on $\mathbb{R}^d$, as $k \to \infty$ we have

$$\int_{\hat\Omega_k} f(x) \, dx = \int_{O_k} f(c_k y) \, dy \, c_k^d \to \int_{\Omega_1} f(y) \, dy.$$

□

6. Relaxed least-action principles for two-fluid incompressible flow and displacement interpolation

In a series of papers that includes [5, 7, 8, 9, 10, 11], Brenier studied Arnold's least-action principles for incompressible Euler flows by introducing relaxed versions that involve convex minimization problems, for which duality principles yield information about minimizers and/or minimizing sequences.

In this section, we describe a simple variant of Brenier's theories that provides a relaxed least-action principle for a two-fluid incompressible flow in which one fluid can be taken as vacuum. For this degenerate case we show that the displacement interpolant (Wasserstein geodesic) provides the unique minimizer. Moreover, the 'incompressible transport sprays' that we constructed in section 5 provide a minimizing sequence for the relaxed problem.

We remark that Lopes Filho et al. [32] studied a variant of Brenier's relaxed least-action principles for variable density incompressible flows. As we indicate below, their formulation is closely related to ours, but it requires fluid density to be positive everywhere.

6.1. **Kinetic energy and least-action principle for two fluids.** We recall that a key idea behind Brenier's work is that kinetic energy can be reformulated in terms of convex duality, based on the idea that kinetic energy is a jointly convex function of density and momentum. In order to handle possible vacuum, we extend this idea in the following way. Let $\hat\varrho \ge 0$ be a constant (representing the density of one fluid). We define $\hat K_{\hat\varrho}$ as the Legendre transform of the indicator function of the paraboloid

$$P_{\hat\varrho} = \{(a,b) \in \mathbb{R}\times\mathbb{R}^d : a + \frac{1}{2}\hat\varrho|b|^2 \le 0\}, \tag{6.1}$$

given for $(x,y) \in \mathbb{R}\times\mathbb{R}^d$ by

$$\hat K_{\hat\varrho}(x,y) = \sup_{(a,b)\in P_{\hat\varrho}} ax + b\cdot y. \tag{6.2}$$

We find the following.

Lemma 6.1. *Let $\hat\varrho \ge 0$ and define $\hat K$ by (6.2). Then $\hat K_{\hat\varrho}$ is convex, and*

$$\hat K_{\hat\varrho}(x,y) = \begin{cases} \dfrac{1}{2}\dfrac{|y|^2}{\hat\varrho x} & \text{if } y \ne 0 \text{ and } \hat\varrho x > 0, \\ 0 & \text{if } y = 0 \text{ and } x \ge 0, \\ +\infty & \text{else.} \end{cases} \tag{6.3}$$

In case $\hat\varrho > 0$, we have the scaling property

$$\hat K_{\hat\varrho}(\hat\varrho x, \hat\varrho y) = \hat K_1(x,y). \tag{6.4}$$

The proof of this lemma is a straightforward calculation based on cases that we leave to the reader. We emphasize that $\hat\varrho = 0$ is allowed. Indeed, for $\hat\varrho = 0$, $\hat K_0$ reduces to the indicator function for the closed half-line

$$\{(x,y) : y = 0,\ x \ge 0\}.$$

Suppose $c \in \mathbb{R}$ represents the 'concentration' of one fluid and $m \in \mathbb{R}^d$ represents the 'momentum' of this fluid, at some point in the flow. If $\hat K_{\hat\varrho}(c,m) < +\infty$, then $c \ge 0$ and $m = \hat\varrho c v$ for some 'velocity' $v \in \mathbb{R}^d$ which satisfies

$$\hat K_{\hat\varrho}(c,m) = \frac{1}{2}\hat\varrho c|v|^2. \tag{6.5}$$

Next we begin to describe our relaxed least-action principle for two-fluid incompressible flow. Consider fluid flow inside a large box for unit time, with

$$\Omega = [-L, L]^d, \qquad Q = \Omega \times [0,1].$$

Let $\hat\varrho_i$, $i = 0, 1$, be constants representing the densities of two fluids, with $\hat\varrho_1 > \hat\varrho_0 \geq 0$. (More fluids could be considered, but we have no reason to do so at this point.) Next we let $c_i(x,t)$, $i = 0, 1$, represent the concentration of fluid i at the point $(x,t) \in Q$. For classical flows, the fluids should occupy non-overlapping regions of space-time, meaning that the concentrations are characteristic functions $c_i = \mathbb{1}_{Q_i}$ with

$$Q_i = \bigcup_{t\in[0,1]} \Omega_{i,t} \times \{t\}, \qquad Q = \bigsqcup_i Q_i. \tag{6.6}$$

The requirement $c_i(x,t) \in \{0,1\}$ will be relaxed, however, to the requirement $c_i(x,t) \in [0,1]$. This provides a convex restriction that heuristically allows 'mixtures' to form (by taking weak limits, say).

Writing $m_i(x,t)$ for the momentum of fluid i at $(x,t) \in Q$, the action to be minimized is the total kinetic energy

$$K(c,m) = \sum_i \int_Q \hat K_{\hat\varrho_i}(c_i, m_i)\, dx\, dt, \tag{6.7}$$

subject to three types of constraints—incompressibility, transport that conserves the total mass of each fluid, and endpoint conditions. We require

$$\sum_i c_i = 1 \qquad \text{a.e. in } Q, \tag{6.8}$$

$$\hat\varrho_i \partial_t c_i + \nabla\cdot m_i = 0 \qquad \text{in } Q \text{ for all } i, \tag{6.9}$$

$$\frac{d}{dt}\int_\Omega \hat\varrho_i c_i = 0 \qquad \text{for } t \in [0,1] \text{ for all } i, \tag{6.10}$$

and fixed endpoint conditions at $t = 0, 1$:

$$c_i(x,0) = \mathbb{1}_{\Omega_{i,0}} \qquad c_i(x,1) = \mathbb{1}_{\Omega_{i,1}}, \tag{6.11}$$

where $\Omega_{i,0}$, $\Omega_{i,1}$ are prescribed for each i.

These constraints are more properly written and collected in the following weak form, required to hold for all test functions p, ϕ_i in the space $C^0(Q)$ of continuous functions on Q, having $\partial_t\phi_i$, $\nabla_x\phi_i$ also continuous on Q, for $i = 0, 1$:

$$\begin{aligned} 0 = \int_Q p - \sum_i \int_Q \big((p + \hat\varrho_i\partial_t\phi_i)\, c_i + \nabla_x\phi_i \cdot m_i\big) \\ - \sum_i \hat\varrho_i \left(\int_{\Omega_{i,1}} \phi_i(x,1)\, dx - \int_{\Omega_{i,0}} \phi_i(x,0)\, dx \right). \end{aligned} \tag{6.12}$$

Let us now describe precisely the set $\mathcal{A}_K$ of functions (c,m) that we take as admissible for the relaxed least-action principle. We require $c_i \in L^\infty(Q,[0,1])$. As we shall see below, it is natural to require that the path

$$t \mapsto c_i(\cdot,t)\, dx$$

is weak-$\star$ continuous into the space of signed Radon measures on Ω, and that $m_i = \hat\varrho_i c_i v_i$ with $v_i \in L^2(Q, c_i)$ if $\hat\varrho_i > 0$. Then the action in (6.7) becomes

$$K(c,m) = \sum_i \int_Q \frac{1}{2}\hat\varrho_i c_i |v_i|^2 . \tag{6.13}$$

When $\hat\varrho_0 = 0$, we require $m_0 = 0$ a.e., since this condition is necessary to have $K(c,m) < \infty$ in (6.7). In this case we have

$$K(c,m) = \int_Q \frac{1}{2}\hat\varrho_1 c_1 |v_1|^2 , \tag{6.14}$$

and the constraints on c_0 from (6.12) reduce simply to the requirement that $c_0 = 1 - c_1$.

We let $\mathcal{A}_K$ denote the set of functions (c,m) that have the properties required in the previous paragraph and satisfy the weak-form constraints (6.12). Our *relaxed least-action two-fluid problem* is to find $(\bar c, \bar m) \in \mathcal{A}_K$ with

$$K(\bar c, \bar m) = \inf_{(c,m)\in\mathcal{A}_K} K(c,m). \tag{6.15}$$

A formal description of classical critical points of the action in (6.15), subject to the constraints in (6.12), and with each c_i a characteristic function of smoothly deforming sets as in (6.6), will lead to classical Euler equations for two-fluid incompressible flow, along the lines of our calculation in section 3, which applies in the case $\hat\varrho_0 = 0$.

We will discuss in subsection 6.3 below how the least-action problem (6.15) is equivalent to a weaker formulation in which (c_i, m_i) are only taken to be signed Radon measures on Q. When $\hat\varrho_0 > 0$, this weaker formulation may be compared directly to the variant of Brenier's least-action principle for variable-density flows, as treated by Lopes Filho et al. [32], in the two-fluid special case.

6.2. **Wasserstein geodesics are minimizers of relaxed action.** We focus now on the case $\hat\varrho_0 = 0$, and take $\hat\varrho_1 = 1$ for convenience.

Theorem 6.2. *Suppose $\hat\varrho_0 = 0$, and Ω_0, Ω_1 are open sets with equal volume and with compact closure in $(-L,L)^d$. Then the relaxed least-action problem in* (6.15)*, with $\Omega_{1,t} = \Omega_t$ for $t = 0,1$, has a unique solution $(\bar c, \bar m)$ given inside Q by*

$$\bar c_1 = \rho, \quad \bar m_1 = \rho v, \qquad \bar c_0 = 1-\rho, \quad \bar m_0 = 0, \tag{6.16}$$

in terms of the displacement interpolant (ρ, v) (described in section 2) between the measures μ_0 and μ_1 with densities $\mathbb{1}_{\Omega_0}$ and $\mathbb{1}_{\Omega_1}$.

Proof. It is clear from the description of section 2 that $(\bar c, \bar m)$ as defined in (6.16) belongs to the admissible set $\mathcal{A}_K$, due to the facts that (i) $0 \le \rho \le 1$ by Lemma 2.1 and (ii) the support of (ρ, v) is compactly contained in Ω due to (2.1). We then have, since $\hat\varrho_0 = 0$,

$$K(\bar c, \bar m) = \int_Q \frac{1}{2}\rho|v|^2 = \int_0^1 \int_{\mathbb{R}^d} \frac{1}{2}\rho |v|^2 \, dx \, dt$$

because the pair (ρ, v) is defined on $\mathbb{R}^d \times [0,1]$ and is zero outside Q. But similarly, for *any* admissible pair $(c,m) \in \mathcal{A}_K$, if we extend (c_1, v_1) by zero outside Q, we have

$$K(c,m) = \int_0^1 \int_{\mathbb{R}^d} \frac{1}{2} c_1 |v_1|^2 \, dx \, dt$$

and (c_1, v_1) determines a narrowly continuous path of measures $t \mapsto \mu_t = c_1 \, dx$ on $\mathbb{R}^d$ with $v \in L^2(\mu)$ that satisfies the continuity equation. It is known that (ρ, v) minimizes this expression over this wider class of paths of measures, due to the characterization of Wasserstein distance by Benamou and Brenier [4], see [45, Thm. 8.1]. By consequence we obtain that $(\bar{c}, \bar{m})$ as defined by (6.16) is indeed a minimizer of the relaxed least-action problem (6.15).

Because the Wasserstein minimizing path is unique (as discussed in section 2), it follows that any minimizer in (6.15) must be given as in (6.16). □

Proposition 6.3. *The family of incompressible flows given for all small $\varepsilon > 0$ by Theorem 5.2 determine a minimizing family $(c^\varepsilon, m^\varepsilon)$ for the relaxed least-action principle* (6.15) *according to*

$$c_1^\varepsilon = \mathbb{1}_{O_t}, \quad m_1^\varepsilon = \mathbb{1}_{O_t} u, \qquad c_0^\varepsilon = 1 - \mathbb{1}_{O_t}, \quad m_0^\varepsilon = 0.$$

That is, $(c^\varepsilon, m^\varepsilon) \in \mathcal{A}_K$ *and* $\lim_{\varepsilon \to 0} K(c^\varepsilon, m^\varepsilon) = \inf_{\mathcal{A}_K} K(c, m)$.

We remark that we are not able to use the Euler sprays that we construct for the proof of Theorem 1.2 to obtain a similar result. The reason is that the target set $\Omega_{1,1} = \Omega_1$ is not hit exactly by our Euler sprays, and this means that the corresponding concentration-momentum pair $(c^\varepsilon, m^\varepsilon) \notin \mathcal{A}_K$ because it would not satisfy the constraint (6.12) as required. We conjecture, however, that for small enough $\varepsilon > 0$, Euler sprays can be constructed that hit an arbitrary target shape Ω_1 (up to a set of measure zero). If that is the case, these Euler sprays would similarly provide a minimizing family for the relaxed least-action principle (6.15).

6.3. **Extended relaxed least-action principle.** In this subsection we discuss an extenstion of the least-action principle (6.15) which facilitates comparison with previous works. Our extension involves expanding the class of admissible concentration-momentum pairs, and is a kind of hybrid of Brenier's 'homogenized vortex sheet' formulation in [8] and the variable-density formulation in [32] for geodesic flow in the diffeomorphism group. The extended formulation reduces, however, to the formulation in (6.15) whenever the action is finite—see Proposition 6.5 below.

The formulations of [8, 9, 32] were designed to make it possible to establish existence of minimizers through convex analysis. The key is to express kinetic energy through duality. We start with the space $C^0(Q)$ of continuous functions on $Q = [-L, L]^d \times [0, 1]$, whose dual is the space $\mathcal{M}(Q)$ of signed Radon measures. The duality pairing is

$$\langle F, c \rangle = \int_Q F\, dc \qquad \text{for } F \in C^0(Q),\ c \in \mathcal{M}(Q).$$

Similarly we write $\langle G, m \rangle = \int_Q G \cdot dm$ for $G \in C^0(Q)^d$ and $m \in \mathcal{M}(Q)^d$.

Next, let $\hat{\varrho} \geq 0$ be a constant representing fluid density. We let

$$\hat{E} = C^0(Q) \times C^0(Q)^d, \quad \hat{E}^* = \mathcal{M}(Q) \times \mathcal{M}(Q)^d,$$

and define $\hat{\mathcal{K}}_{\hat{\varrho}} : \hat{E}^* \to \mathbb{R}$ as the Legendre transform of the indicator function of the parabolic set

$$\mathcal{P}_{\hat{\varrho}} = \{(F, G) \in E : F + \frac{1}{2}\hat{\rho}|G|^2 \leq 0 \ \text{ in } Q\}, \tag{6.17}$$

given for $(c, m) \in \hat{E}^*$ by

$$\hat{\mathcal{K}}_{\hat{\varrho}}(c, m) = \sup_{(F,G) \in \mathcal{P}_{\hat{\varrho}}} \langle F, c \rangle + \langle G, m \rangle. \tag{6.18}$$

(To compare with [32, eq. (3.8)] it may help to note $\hat{\varrho}\mathcal{P}_{\hat{\varrho}} = \mathcal{P}_1$ when $\hat{\varrho} > 0$.)

The following result follows from [8, Proposition 3.4] in the case $\hat{\varrho} > 0$, and is straightforward to show in the case $\hat{\varrho} = 0$, when the conclusion entails $m = 0$.

Proposition 6.4. *Let $\hat\varrho \geq 0$, and let $(c,m) \in \hat E^*$. If $\hat K_{\hat\varrho}(c,m) < \infty$, then c is a nonnegative measure and m is absolutely continuous with respect to c, with Radon-Nikodým derivative $\hat\varrho v$ where $v \in L^2(Q,c)$, and*

$$\hat{\mathcal{K}}_{\hat\varrho}(c,m) = \int_Q \frac{1}{2}\hat\varrho|v|^2\,dc.$$

Our reformulated least-action problem may now be specified, as follows. Let $\hat\varrho_1 > \hat\varrho_0 \geq 0$. For $(c,m) \in E^* = \hat E^* \times \hat E^*$ we write

$$c = (c_0, c_1), \quad m = (m_0, m_1),$$

and we define

$$\mathcal{K}(c,m) = \sum_i \mathcal{K}_{\hat\varrho_i}(c_i, m_i). \tag{6.19}$$

We introduce the class $\hat{\mathcal{A}}_{\mathcal{K}}$ of admissible pairs $(c,m) \in E^*$ that satisfy the same weak-form constraints (6.12) as before (with c_i, m_i replaced respectively by dc_i, dm_i). The extended relaxed least-action problem is to find $(\hat c, \hat m) \in \hat{\mathcal{A}}_{\mathcal{K}}$ such that

$$\mathcal{K}(\hat c, \hat m) = \inf_{(c,m)\in\hat{\mathcal{A}}_{\mathcal{K}}} \mathcal{K}(c,m). \tag{6.20}$$

This form of the relaxed least-action problem may be compared rather directly with the homogenized vortex sheet model of Brenier [8] and with the variable-density model of Lopes Filho et al. [32]. Both of these models deal with the endpoint problem for diffeomorphisms rather than mass distributions as is done here. Brenier's model involves a sum over 'phases' as in our model (6.19), but the fluid density in each phase is the same. The variable-density model of [32] allows for mixture density (called c, corresponding to $\hat\varrho c$ here) to depend upon both Eulerian and Lagrangian spatial coordinates (called x and a respectively), similar to the formulation in [9].

In both [8] and [32] as well as related works for relaxed least-action principles formulated in a space of measures, the existence of minimizers is established by using the Fenchel-Rockafellar theorem from convex analysis. One expresses the objective function corresponding to $\mathcal{K}(c,m)$ as a sum of Legendre transforms of indicator functions of two sets, corresponding here to the set $\mathcal{P}_{\hat\varrho}$ in (6.17) and to another set that accounts for the constraints in (6.12). We do not pursue this analysis as it is outside the scope of this paper. In any case, for the degenerate case $\hat\varrho_0 = 0$ that is most relevant to the rest of this paper, existence of a unique minimizer follows from Theorem 6.2 above and Proposition 6.5 below.

We claim that the relaxed least-action problem (6.20) always reduces to the previous problem (6.15), due to the following fact.

Proposition 6.5. *Suppose $(c,m) \in \hat{\mathcal{A}}_{\mathcal{K}}$ and $\mathcal{K}(c,m) < \infty$. Then for some $(\bar c, \bar m) \in \mathcal{A}_K$ we have $\mathcal{K}(c,m) = K(\bar c, \bar m)$ and*

$$dc_i = \bar c_i\,dx\,dt, \quad dm_i = \bar m_i\,dx\,dt, \quad i = 0, 1. \tag{6.21}$$

Consequently, the infimum in (6.20) is the same as that in (6.15).

Proof. To prove this result, we first invoke Proposition 6.4 to infer that c_i is a nonnegative measure and m_i is absolutely continuous with respect to c_i for $i = 0, 1$. Next we note that $\sum_i c_i = 1$ by taking $\phi_i = 0$ and p arbitrary in (6.12). Hence the representation in (6.21) holds with $\bar c_i \in L^\infty(Q, [0,1])$ and $m_i = \hat\varrho_i \bar c_i v_i$ with $v_i \in L^2(Q, \bar c_i)$.

Finally, we claim $t \mapsto \bar c_i(\cdot, t)$ is weak-$\star$ continuous into $\mathcal{M}(Q)$. By choosing $p = 0$ and ϕ_i to depend only on t in (6.12) we infer that $\int_\Omega \bar c_i(x,t)\,dx$ is independent of t. Thus, because Ω

is compact, we can invoke Lemma 8.1.2 of [2] to conclude that $t \mapsto \bar{c}_i(\cdot, t)$ is narrowly, hence weak-$\star$, continuous.

It is clear that the infimum in (6.15) is greater or equal to that in (6.20), because the admissible set $\mathcal{A}_K$ is naturally embedded in $\hat{\mathcal{A}}_{\mathcal{K}}$, and the two are equal if either is finite. Recalling that $\inf \emptyset = +\infty$, equality follows in general. $\square$

Remark 6.6. As a last comment, we note that for variable-density flow with strictly positive density, the relaxed least-action problem studied by Lopes et al. [32] was shown to be *consistent* with the classical Euler equations, in the sense that classical solutions of the Euler system induce weak solutions of relaxed Euler equations, and for sufficiently short time the induced solution is the unique minimizer of the relaxed problem. In the case that we consider with $\hat{\rho}_0 = 0$, however, this consistency property is unlikely to hold in general when the space dimension $d > 1$, for the reason that in general we can expect the Wasserstein density $\rho < 1$ in Theorem 6.2, while necessarily $\rho \in \{0, 1\}$ for any classical solution of the incompressible Euler equations.

7. Extensions

Theorem 1.1 establishes that restricting the Wasserstein metric to paths of shapes of fixed volume does not provide a new notion of distance on the space of such shapes. Namely it shows that for paths in the space of shapes of fixed volume, the infimum of the length of paths between two given shapes is the Wasserstein distance.

Volume change. It is of interest to consider a more general space of shapes in order to compare shapes of different volumes. In particular, the Schmitzer and Schnörr [41] considered a space that corresponds to the set of bounded, simply connected domains in $\mathbb{R}^2$ with smooth boundary and arbitrary positive area. To each such shape Ω one associates as its corresponding *shape measure* the probability measure having uniform density on Ω, denoted by

$$\mathcal{U}_\Omega = \frac{1}{|\Omega|} \mathbb{1}_\Omega. \tag{7.1}$$

We consider here this same association between sets and shape measures, but allow for more general shapes. Namely for fixed dimension d, let us consider shapes as bounded measurable subsets of $\mathbb{R}^d$ with positive volume. Let $\mathcal{C}$ be the set of all shape measures corresponding to such shapes. Thus $\mathcal{C}$ is the set of all uniform probability distributions of bounded support.

One can formally consider $\mathcal{C}$ as a submanifold of the space of probability measures of finite second moment, endowed with Wasserstein distance. Then we define a distance between shapes as we did in (1.1):

$$d_{\mathcal{C}}(\Omega_0, \Omega_1) = \inf \mathcal{A}\,, \qquad \mathcal{A} = \int_0^1 \int_{\mathbb{R}^d} \rho |v|^2 \, dx \, dt\,, \tag{7.2}$$

where $\rho = (\rho_t)$ is now required to be a path of shape measures in $\mathcal{C}$, with endpoints

$$\rho_0 = \mathcal{U}_{\Omega_0}\,, \qquad \rho_1 = \mathcal{U}_{\Omega_1}\,, \tag{7.3}$$

and transported according to the continuity equation (1.2) with a velocity field $v \in L^2(\rho \, dx \, dt)$.

Because the characteristic-function restriction (1.4) is replaced by the weaker requirement that ρ_t has a uniform density, for any two shapes of equal volume scaled to unity for convenience, it is clear that

$$d_s(\Omega_0, \Omega_1) \geq d_{\mathcal{C}}(\Omega_0, \Omega_1) \geq d_W(\mathbb{1}_{\Omega_0}, \mathbb{1}_{\Omega_1}). \tag{7.4}$$

Then as a direct consequence of Theorem 1.1, we have

$$d_{\mathcal{C}}(\Omega_0, \Omega_1) = d_W\left(\mathbb{1}_{\Omega_0}, \mathbb{1}_{\Omega_1}\right). \tag{7.5}$$

By a minor modification of the arguments of section 5, in general we have the following.

Theorem 7.1. *Let Ω_0 and Ω_1 be any two shapes of positive volume. Then*

$$d_C(\Omega_0, \Omega_1) = d_W(\mathcal{U}_{\Omega_0}, \mathcal{U}_{\Omega_1}).$$

The proof, provided in subsection 7.1 below, makes use of a displacement convexity argument closely related to McCann's well-known proof of the Brunn-Minkowski inequality using mass transportation, see [45, p. 186].

Smoothness. For dimension $d = 2$, Theorem 7.1 does not apply to describe distance in the space of shapes considered by Schmitzer and Schnörr in [41], however, for as we have mentioned, they consider shapes to be bounded simply connected domains with smooth boundary.

One point of view on this issue is that it is nowadays reasonable for many purposes to consider 'pixelated' images and shapes, made up of fine-grained discrete elements, to be valid approximations to smooth ones. Thus the microdroplet constructions considered in this paper, which fit with the mathematically natural regularity conditions inherent in the definition of Wasserstein distance, need not be thought unnatural from the point of view of applications.

Nevertheless one may ask whether the infimum of path length in the space of smooth simply connected shapes is still the Wasserstein distance, as in Theorem 7.1. Our proof of Theorem 1.1 in Section 5 does not provide paths in this space because the union of droplets is disconnected. However, the main mechanism by which we efficiently transport mass, namely by "dividing" the domain into small pieces (droplets) which almost follow the Wasserstein geodesics, is still available. In particular, by creating many deep creases in the domain it might be effectively 'divided' into such pieces while still remaining connected and smooth. Thus we conjecture that even in the class of smooth sets considered in [41], the geodesic distance is the Wasserstein distance between uniform distributions as in Theorem 7.1.

Geodesic equations. It is also interesting to compare our Euler droplet equations from subsection 3.1 with the formal geodesic equations for smooth critical paths of the action $\mathcal{A}$ in the space $\mathcal{C}$ of uniform distributions. These equations correspond to equation (4.12) of Schmitzer and Schnörr in [41].

These geodesic equations may be derived in a manner almost identical to the treatment in subsection 3.1 above. The principal difference is that due to (3.4), the divergence of the Eulerian velocity may be a nonzero function of time, constant in space:

$$\nabla \cdot v = c(t),$$

and the same is true of virtual displacements $\tilde{v}$. The variation of action now satisfies

$$\frac{\delta \mathcal{A}}{2} = \int_{\Omega_t} v \cdot \tilde{v}\, \rho\, dx \bigg|_{t=1} - \int_0^1 \int_{\Omega_t} (\partial_t v + v \cdot \nabla v) \cdot \tilde{v}\, \rho\, dx\, dt. \tag{7.6}$$

Now, the space of vector fields orthogonal to all constant-divergence fields on Ω_t is the space of gradients ∇p such that p vanishes on the boundary and has average zero in Ω_t, satisfying

$$p = 0 \quad \text{on } \partial\Omega_t, \qquad \int_{\Omega_t} p\, dx = 0. \tag{7.7}$$

Because ρ is spatially constant and $\tilde{v}$ can be (locally in time) arbitrary with spatially constant divergence, necessarily $u = -(\partial_t v + v \cdot \nabla v)$ is such a gradient. The remaining considerations in section 3.1 apply almost without change, and we conclude that $v = \nabla\phi$ where

$$\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + p = 0, \qquad \Delta\phi = c(t), \tag{7.8}$$

where $c(t)$ is spatially constant in Ω_t.

These fluid equations differ from those in section 3.1 in that ϕ gains one degree of freedom (a multiple of the solution of $\Delta\phi = 1$ in Ω_t with Dirichlet boundary condition) while the pressure p loses one degree of freedom (as its integral is constrained).

They have elliptical droplet solutions given by displacement interpolation of elliptical Wasserstein droplets as in subsection 3.4, because pressure vanishes and density is indeed spatially constant for these interpolants. Because they are Wasserstein geodesics, these particular solutions are also length-minimizing geodesics in the shape space $\mathcal{C}$. It seems likely that length minimizing paths in $\mathcal{C}$ will not generally exist even locally, but we have no proof at present.

7.1. **Proof of Theorem 7.1.** We assume at first that the source Ω_0 and target Ω_1 are bounded open sets. (This extends to the general case of measurable sets as before in subsection 5.3.) The general idea of the proof is to make use of the 'incompressible' smooth sprays constructed in section 5, dilating their flow maps in a suitable way.

We may assume that Ω_0 has unit volume without loss of generality. As in section 2, the displacement interpolant provides the Wasserstein geodesic path between the uniform densities $\mathcal{U}_{\Omega_0}$ and $\mathcal{U}_{\Omega_1}$, with straight-line particle paths coming from the Brenier map $T = \nabla\psi$ via

$$T_t(x) = (1-t)x + tT(x)\,.$$

Along each particle path starting in the non-singular set $\Omega_0 \setminus \Sigma_0$, the density ρ is smooth and the function $\rho^{-1/d}$ is concave, just as shown in Lemma 2.1, satisfying

$$\rho(T_t(x), t)^{-1/d} \geq (1-t)\rho_0^{-1/d} + t\rho_1^{-1/d} =: b(t). \tag{7.9}$$

Let $\varepsilon > 0$ and $D = d_W(\mathcal{U}_{\Omega_0}, \mathcal{U}_{\Omega_1})^2$. We will cover the non-singular set up to a null set by a disjoint union of balls

$$O_0 = \bigsqcup_i O_{i,0}, \qquad O_{i,0} := B(x_i, r_i), \tag{7.10}$$

determined by a Vitali covering argument in the same way as at the beginning of the proof of Theorem 5.2 so that (5.9) holds. Just as shown at the end of the proof of Proposition 5.3, the images $T_t(O_{i,0})$ remain star-shaped with respect to $T_t(x_i)$ for all $t \in [0,1]$.

To create a path in the shape space, we will rearrange the mass pushed forward by T_t from $O_{i,0}$ to form a patch supported on a subset of the image $T_t(O_{i,0})$, with constant density $\tilde{\rho}_t$ the same for all balls. To do this, we first dilate $T_t(O_{i,0})$ about the point $T_t(x_i)$ to maintain the key property that its expansion factor, the ratio of its volume at time t to that at time 0, is independent of i. We will set

$$S_t(x) = T_t(x_i) + a_i(t)(T_t(x) - T_t(x_i)) \qquad \text{for } x \in B(x_i, r_i), \tag{7.11}$$

where we desire $a_i(0) = a_i(1) = 1$ and $0 < a_i(t) \leq 1$ to ensure that

$$O_{i,t} := S_t(O_{i,0}) \subset T_t(O_{i,0}). \tag{7.12}$$

This is needed to guarantee the dilated images $O_{i,t}$ remain disjoint.

To see how to determine $a_i(t)$, we recall that the uniform measures on Ω_0 and Ω_1 have respective densities $\rho_0 = |\Omega_0|^{-1} = 1$ and $\rho_1 = |\Omega_1|^{-1}$. Because we have

$$\int_{T_t(O_{i,0})} \rho(z,t)\,dz = \int_{O_{i,0}} dx = |O_{i,0}|$$

by the change of variables $z = T_t(x)$, integration of (7.9) and use of Hölder's inequality yields

$$\begin{aligned} b(t) &\le \int_{O_{i,0}} \rho(T_t(x),t)^{-1/d} \frac{dx}{|O_{i,0}|} = \int_{T_t(O_{i,0})} \rho(z,t)^{-1/d} \frac{\rho(z,t)\,dz}{|O_{i,0}|} \\ &\le \left(\int_{T_t(O_{i,0})} \frac{1}{\rho(z,t)} \frac{\rho(z,t)\,dz}{|O_{i,0}|} \right)^{1/d} = \left(\frac{|T_t(O_{i,0})|}{|O_{i,0}|} \right)^{1/d} \end{aligned} \tag{7.13}$$

with left-hand side independent of i.

Next, similar to the definition in (5.2), define

$$\tilde{S}_t(x) = T_t(x_i) + \tilde{a}_i(t)(T_t(x) - T_t(x_i)) \qquad \text{for } x \in B(x_i, r_i), \tag{7.14}$$

where

$$\tilde{a}_i(t) = \left(\frac{|T_t(O_{i,0})|}{|O_{i,0}|} \right)^{-1/d}.$$

As in section 5, this scaling preserves the volume of ball images: With $\tilde{O}_{i,t} = \tilde{S}_t(O_{i,0})$ we have $|\tilde{O}_{i,t}| = |O_{i,0}|$ for all $t \in [0,1]$ and all i. (These images may overlap, but we will consider them separately below.) We choose $a_i(t)$ to satisfy

$$a_i(t) = b(t)\tilde{a}_i(t). \tag{7.15}$$

We indeed find $a_i(0) = a_i(1) = 1$ and $0 < a_i(t) \le 1$ due to (7.13), and also that the ball images dilated by S_t satisfy

$$|O_{i,t}| = b(t)^d \tilde{a}_i(t)^d |T_t(O_{i,0})| = b(t)^d |O_{i,0}| \tag{7.16}$$

with expansion factor $b(t)^d$ independent of i.

We follow the procedure in section 5.1 to modify the velocity field for the flow $\tilde{S}_t$ on each individual ball and obtain divergence-free velocity fields $\tilde{u}_i$ on the space-time domains

$$\tilde{Q}_i = \bigcup_{t\in[0,1]} \tilde{O}_{i,t} \times \{t\}.$$

The velocity field $\tilde{u}_i$ determines a Lagrangian flow map $\tilde{X}_i(x,t)$ on $\tilde{O}_{i,0} \times [0,1]$ satisfying

$$\partial_t \tilde{X}_i(x,t) = \tilde{u}_i(\tilde{X}_i(x,t),t), \qquad \tilde{X}_i(O_{i,0},t) = \tilde{O}_{i,t} \quad \text{for all } t \in [0,1] \text{ and all } i.$$

This map is volume-preserving with $\det \partial\tilde{X}_i/\partial x \equiv 1$ in its domain. Because

$$\tilde{X}_i(x,t) = x + \int_0^t \tilde{u}(\tilde{X}_i(x,s))\,ds, \qquad T_t(x_i) = x_i + tv_0(x_i),$$

due to estimate (5.7) from Proposition 5.3 we find

$$\|\tilde{X}_i(\cdot,t) - T_t(x_i)\|_{O_{0,i}} \le \|x - x_i\|_{O_{0,i}} + \int_0^t \|\tilde{u}(\tilde{X}_i(\cdot,s)) - v_0(x_i)\|_{O_{0,i}} \le C_i r_i^{(d+2)/2}. \tag{7.17}$$

Here and below C_i denote constants $C(\underline{\lambda}_{r_i}, \overline{\lambda}_{r_i})$, whose values are used to determine the Vitali covering as in subsection 5.2 according to (5.22).

Finally we can define our dilated flow map on $O_0 \times [0,1]$ via

$$X(x,t) = T_t(x_i) + b(t)(\tilde{X}_i(x,t) - T_t(x_i)), \qquad x \in B(x_i, r_i). \tag{7.18}$$

This flow map has spatially constant Jacobian $\det \partial X/\partial x = b(t)^d$ the same on all balls. Moreover, this map carries the ball $O_{i,0} = B(x_i, r_i)$ to the set

$$\begin{aligned} X(O_{i,0}, t) &= T_t(x_i) + b(t)(\tilde{O}_{i,t} - T_t(x_i)) \\ &= T_t(x_i) + b(t)\tilde{a}_i(t)(T_t(O_{i,0}) - T_t(x_i)) \\ &= O_{i,t} \subset T_t(O_{i,0}) \end{aligned} \tag{7.19}$$

The last inclusion is due to (7.12) and implies that these images remain disjoint for all $t \in [0,1]$.

The flow map X induces an Eulerian velocity field u on the space-time domain

$$Q = \bigsqcup_i O_{i,t} \times \{t\}$$

with spatially constant divergence. The density $\rho = b^{-d}\mathbb{1}_Q$ is a probability density for each time t, and is transported by u according to the continuity equation.

Thus we get an admissible path of shape measures in the shape space $\mathcal{C}$. The action in (7.2) takes the following form after pulling back using Lagrangian variables:

$$\mathcal{A} = \sum_i \int_0^1 \int_{O_{i,0}} |\partial_t X(x,t)|^2 \, dx \, dt = \sum_i \int_0^1 \|\partial_t X(\cdot,t)\|^2_{O_{i,0}} \, dt \,. \tag{7.20}$$

Note that by (7.18) and the uniform boundedness of $b(t)$ and $b'(t)$,

$$\begin{aligned} \|\partial_t X - v_0(x_i)\|_{O_{i,0}} &\le |b'(t)| \|\tilde{X}_i(\cdot,t) - T_t(x_i)\|_{O_{i,0}} + |b(t)| \|\tilde{u}_i(\tilde{X}_i(\cdot,t)) - v_0(x_i)\|_{O_{i,0}} \\ &\le C_i r_i^{(d+2)/2} \end{aligned} \tag{7.21}$$

Because

$$D := d_W(\mathcal{U}_{\Omega_0}, \mathcal{U}_{\Omega_1})^2 = \int_{\Omega_0} |v_0(x)|^2 \, dx = \sum_i \|v_0\|^2_{O_{i,0}}$$

and $\|v_0 - v_0(x_i)\|_{O_{i,0}} \le C_i r_i^{(d+2)/2}$, we find

$$\begin{aligned} \mathcal{A} &\le \sum_i \left(\|v_0\|_{O_{i,0}} + C_i r_i^{(d+2)/2} \right)^2 \\ &\le \sum_i \left((1+r_i)\|v_0\|^2_{O_{i,0}} + C_i r_i^{d+1} \right) \le D + \varepsilon \end{aligned} \tag{7.22}$$

as in the proof of Theorem 5.2.

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