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GLOBAL WELL-POSEDNESS OF DISSIPATIVE QUASI-GEOSTROPHIC EQUATIONS IN CRITICAL SPACES

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Abstract. We prove global well-posedness for the dissipative quasi-geostrophic equation with initial data in critical Besov spaces $B_{p,q}^{1+\frac{2}{p}-2\alpha}$, $0 < \alpha \leq 1$, provided that the $B_{p,q}^{1+\frac{2}{p}-2\alpha}$ norm of the initial data is sufficiently small compared with the dissipative coefficient κ.

1. Introduction

We are concerned with the two dimensional dissipative quasi-geostrophic equation

$$\text{(DQG)} \qquad \begin{cases} \theta_t + u\cdot\nabla\theta + \kappa(-\triangle)^\alpha\theta = 0, \\ u = (-R_2\theta, R_1\theta), \\ \theta(x,0) = \theta_0(x). \end{cases}$$

where the scalar θ represents the potential temperature, u is the fluid velocity, and R_1, R_2 are the usual Riesz transform. For the physical background of this equation, one may check [1], [3] and references therein for the details. We solve the open problem given by [1]; namely, with $\theta_0 \in B_{p,q}^{1+\frac{2}{p}-2\alpha}$, for $1 \leq p, q < \infty$, what is the well-posedness of (DQG)? Two crucial estimates were proved in [3], [4], and we use those estimates to get the following result.

Theorem. *There exists a constant $\epsilon_0 > 0$ such that for any $\theta_0 \in B_{p,q}^{1+\frac{2}{p}-2\alpha}$ with $||\theta_0||_{B_{p,q}^{1+\frac{2}{p}-2\alpha}} < \epsilon \leq \epsilon_0$, (DQG) has a unique global solution θ, which belongs to $C([0,\infty); B_{p,q}^{1+\frac{2}{p}-2\alpha})$.*

2. Proof of theorem

Step 1. A priori estimates. Let $\triangle_j$ be the Fourier multiplier given by $\triangle_j f = \Phi_j * f$ $(j = 0, \pm1, \pm2, \cdots)$ where $\Phi_j(\xi)$ is a smooth function localized around $|\xi| = 2^j$ satisfying $\sum_{k=-\infty}^{\infty}\Phi_k = 1$, except for $\xi = 0$. Applying the operator $\triangle_j$ to the first equation of (DQG), we obtain

$$\frac{d}{dt}\triangle_j\theta + \triangle_j(u\cdot\nabla\theta) + \kappa(-\triangle)^\alpha\triangle_j\theta = 0.$$

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Multiplying by $\frac{1}{p}\triangle_j\theta\cdot|\triangle_j\theta|^{p-2}$ in the above equation and then integrating with respect to x, we have

$$\frac{d}{dt}||\triangle_j\theta||^p_{L^p}+\kappa\frac{1}{p}\int(-\triangle)^\alpha\cdot\triangle_j\theta\cdot\triangle_j\theta\cdot|\triangle_j\theta|^{p-2}$$
$$=-\frac{1}{p}\int\triangle_j(u\cdot\nabla\theta)\cdot\triangle_j\theta\cdot|\triangle_j\theta|^{p-2}.$$

Wu [4] proved the following lower bound estimate:

$$\int(-\triangle)^\alpha\cdot\triangle_j\theta\cdot\triangle_j\theta\cdot|\triangle_j\theta|^{p-2}$$
$$\geq C\cdot 2^{2j\alpha}\cdot||\triangle_j\theta||^p_{L^p}.$$

So we obtain that

$$\frac{d}{dt}||\triangle_j\theta||^p_{L^p}+C\cdot\kappa\cdot 2^{2j\alpha}\cdot||\triangle_j\theta||^p_{L^p}$$
$$\leq C\cdot|\int\triangle_j(u\cdot\nabla\theta)\cdot\triangle_j\theta\cdot|\triangle_j\theta|^{p-2}dx|.$$

We decompose $(u\cdot\nabla\theta)$ as a paraproduct. (We obtain estimates of this product term. See the appendix.) Then,

$$\frac{d}{dt}||\triangle_j\theta||^p_{L^p}+C\cdot\kappa\cdot 2^{2j\alpha}\cdot||\triangle_j\theta||^p_{L^p}$$
$$\leq C\cdot 2^{2j\alpha}\cdot 2^{-j(1+\frac{2}{p}-2\alpha)}\cdot a_j||\triangle_j\theta||^{p-1}_{L^p}||\theta||^2_{B^{1+\frac{2}{p}-2\alpha}_{p,q}}.$$

Dividing both sides by $||\triangle_j\theta||^{p-1}_{L^p}$,

$$\frac{d}{dt}||\triangle_j\theta||_{L^p}+C\cdot\kappa\cdot 2^{2j\alpha}\cdot||\triangle_j\theta||_{L^p}$$
$$\leq C\cdot 2^{2j\alpha}\cdot 2^{-j(1+\frac{2}{p}-2\alpha)}\cdot a_j||\theta||^2_{B^{1+\frac{2}{p}-2\alpha}_{p,q}}.$$

By solving the above differential equation of time, we get

$$||\triangle_j\theta(t)||_{L^p}\leq e^{-t2^{2j\alpha}\kappa}\cdot||\triangle_j\theta_0||_{L^p}$$
$$+C\cdot a_j\cdot 2^{2j\alpha}\cdot 2^{-j(1+\frac{2}{p}-2\alpha)}\int_0^t e^{-(t-s)2^{2j\alpha}\kappa}||\theta||^2_{B^{1+\frac{2}{p}-2\alpha}_{p,q}}.$$

By Young's inequality in time,

$$||\triangle_j\theta(t)||_{L^\infty_T L^p}\leq||\triangle_j\theta_0||_{L^p}$$
$$+\frac{C}{\kappa}\cdot a_j\cdot 2^{2j\alpha}\cdot 2^{-j(1+\frac{2}{p}-2\alpha)}\cdot||\theta||^2_{\widetilde{L}^\infty_T B^{1+\frac{2}{p}-2\alpha}_{p,q}}.$$

We note that $||\theta(t)||_{L^p}\leq||\theta_0||_{L^p}$ was proved in [3]. So,

$$||\theta||_{\widetilde{L}^\infty_T B^{1+\frac{2}{p}-2\alpha}_{p,q}}\leq||\theta_0||_{B^{\frac{2}{p}+1-2\alpha}_{p,q}}+\frac{C}{\kappa}||\theta||^2_{\widetilde{L}^\infty_T B^{1+\frac{2}{p}-2\alpha}_{p,q}}.$$

Step 2. Iteration and uniform estimates. Because the bicontinuous constant arising in the above estimate does not depend on time, we will look for a solution $w(x,t) = \theta(x,t) - S_\alpha(t)\theta_0$, instead of looking for a solution $\theta(x,t)$, where $S_\alpha(t)\theta_0 = e^{-\kappa t(-\triangle)^\alpha}\theta_0$. $w(x,t)$ satisfies

$$\begin{aligned}
&w_t + u \cdot \nabla(w + S_\alpha(t)\theta_0) + \kappa(-\triangle)^\alpha w = 0,\\
&u = (-R_2(w + S_\alpha(t)\theta_0), R_1(w + S_\alpha(t)\theta_0)),\\
&w(x,0) = 0.
\end{aligned}$$

We define the following sequences:

$$\begin{aligned}
&w_t^{n+1} + u^n \cdot \nabla(w^{n+1} + S_\alpha(t)\theta_0) + \kappa(-\triangle)^\alpha w^{n+1} = 0,\\
&u^n = (-R_2(w^n + S_\alpha(t)\theta_0), R_1(w^n + S_\alpha(t)\theta_0)),\\
&w^{n+1}(x,0) = 0.
\end{aligned}$$

Similarly to a priori estimates, we have

$$\begin{aligned}
&||w^{n+1}||_{\tilde{L}_T^\infty B_{p,q}^{1+\frac{2}{p}-2\alpha}}\\
&\quad\le \frac{C}{\kappa}||w^n||_{\tilde{L}_T^\infty B_{p,q}^{1+\frac{2}{p}-2\alpha}} \cdot (||w^n||_{\tilde{L}_T^\infty B_{p,q}^{1+\frac{2}{p}-2\alpha}} + ||S_\alpha(t)\theta_0||_{\tilde{L}_T^\infty B_{p,q}^{1+\frac{2}{p}-2\alpha}})\\
&\quad\le \frac{C}{\kappa}||w^n||_{\tilde{L}_T^\infty B_{p,q}^{1+\frac{2}{p}-2\alpha}} \cdot (||w^n||_{\tilde{L}_T^\infty B_{p,q}^{1+\frac{2}{p}-2\alpha}} + ||\theta_0||_{B_{p,q}^{1+\frac{2}{p}-2\alpha}}).
\end{aligned}$$

Let $\epsilon_0 \le \frac{\kappa}{4C}$, and fix η such that $\eta < \epsilon_0$. If $||\theta_0||_{B_{p,q}^{1+\frac{2}{p}-2\alpha}} \le \epsilon < \epsilon_0$, then $||w^n||_{\tilde{L}_T^\infty B_{p,q}^{1+\frac{2}{p}-2\alpha}}$ are uniformly bounded by $||w^n||_{\tilde{L}_T^\infty B_{p,q}^{1+\frac{2}{p}-2\alpha}} \le \eta$.

Step 3. Equations of difference, existence, and uniqueness. Let $\delta w^n = w^n - w^{n-1}$, $\delta u^n = u^n - u^{n-1}$. Then we have the following system of difference equations:

$$\begin{aligned}
&\delta w_t^{n+1} + u^n \cdot \nabla \delta w^{n+1} + \kappa(-\triangle)^\alpha \delta w^{n+1} + \delta u^n \cdot \nabla(w^n + S_\alpha(t)\theta_0) = 0,\\
&u^n = (-R_2(w^n + S_\alpha(t)\theta_0), R_1(w^n + S_\alpha(t)\theta_0)), \delta u^n = (-R_2(\delta w^n), R_1(\delta w^n)),\\
&\delta w^{n+1}(x,0) = 0.
\end{aligned}$$

Then, as before, we get

$$\begin{aligned}
||\delta w^{n+1}||_{\tilde{L}_T^\infty B_{p,q}^{1+\frac{2}{p}-2\alpha}} &\le \frac{C}{\kappa}||\delta w^n||_{\tilde{L}_T^\infty B_{p,q}^{1+\frac{2}{p}-2\alpha}} \cdot (||w^n||_{\tilde{L}_T^\infty B_{p,q}^{1+\frac{2}{p}-2\alpha}} + ||\theta_0||_{B_{p,q}^{1+\frac{2}{p}-2\alpha}})\\
&< \frac{C}{\kappa}||\delta w^n||_{\tilde{L}_T^\infty B_{p,q}^{1+\frac{2}{p}-2\alpha}} \cdot (\eta + \epsilon) < \frac{1}{2} \cdot ||\delta w^n||_{\tilde{L}_T^\infty B_{p,q}^{1+\frac{2}{p}-2\alpha}}.
\end{aligned}$$

So, w^n converges to w in $L_T^\infty B_{p,q}^{1+\frac{2}{p}-2\alpha}$. Furthermore, we can take η as small as we want. Hence w^n converges to w in $C([0,T); B_{p,q}^{1+\frac{2}{p}-2\alpha})$. Uniqueness can be proved similarly. This completes the proof of theorem.

3. Appendix

We decompose $(u\cdot\nabla\theta)$ as a paraproduct:

$$\begin{aligned}\triangle_j(u\cdot\nabla\theta) &= \sum_{|j-l|\le N}\triangle_j(S_{l-1}u\cdot\triangle_l\nabla\theta)+\sum_{|j-l|\le N}\triangle_j(\triangle_l u\cdot S_{l-1}\nabla\theta)\\ &\quad+\sum_{l\ge j-N}\sum_{|l-m|\le 1}\triangle_j(\triangle_l u\cdot\triangle_m\nabla\theta).\end{aligned} \tag{1}$$

So we have three terms to the right-hand side of (1). Motivated by [2], we decompose I defined below as

$$\begin{aligned} I &= \sum_{|l-j|\le N}\left|\int\triangle_j(S_{l-1}u\cdot\triangle_l\nabla\theta)\cdot\triangle_j\theta\cdot|\triangle_j\theta|^{p-2}dx\right|\\ &\le \left|\sum_{|l-j|\le N}\int[\triangle_j,S_{l-1}u]\nabla\triangle_l\theta\cdot\triangle_j\theta|\triangle_j\theta|^{p-2}\right|\\ &\quad+\left|\sum_{|l-j|\le N}\int(S_{l-1}u-S_{j-1}u)\nabla\triangle_j\triangle_l\theta\cdot\triangle_j\theta|\triangle_j\theta|^{p-2}\right|\\ &\quad+\left|\sum_{|l-j|\le N}\int S_{j-1}u\cdot\nabla\triangle_j\triangle_l\theta\cdot\triangle_j\theta\cdot|\triangle_j\theta|^{p-2}dx\right|\\ &= I_1+I_2+I_3.\end{aligned}$$

I_3 disappears when integrated, by the divergence free condition of u. (From now on, we repeatedly use Bernstein's inequalities.) By Hölder inequality,

$$\begin{aligned} I_1 &= \left|\sum_{|l-j|\le N}\int[\triangle_j,S_{l-1}u]\nabla\triangle_l\theta\cdot\triangle_j\theta|\triangle_j\theta|^{p-2}\right|\\ &\le C\cdot\|[\triangle_j,S_{j-1}u]\nabla\triangle_j\theta\|_{L^p}\cdot\|\triangle_j\theta\|_{L^p}^{p-1}\\ &\le C\cdot 2^{-j}\|\nabla S_{j-1}u\|_{L^\infty}\|\nabla\triangle_j\theta\|_{L^p}\|\triangle_j\theta\|_{L^p}^{p-1}.\end{aligned}$$

But, by the Calderon-Zygmund theorem, we have

$$\|\nabla S_{j-1}u\|_{L^\infty}\le C\cdot 2^{2j\alpha}\|\theta\|_{B^{1+\frac{2}{p}-2\alpha}_{p,q}}.$$

Therefore

$$I_1\le C\cdot 2^{2j\alpha}\cdot 2^{-j(1+\frac{2}{p}-2\alpha)}\cdot a_j\|\triangle_j\theta\|_{L^p}^{p-1}\|\theta\|^2_{B^{1+\frac{2}{p}-2\alpha}_{p,q}}$$

where $\{a_j\}\in l^q$ such that $\sum_{j\ge -1}a_j^q=1$. Similarly

$$\begin{aligned} I_2 &\le C\cdot\|\triangle_j u\|_{L^\infty}\|\nabla\triangle_j\theta\|_{L^p}\|\triangle_j\theta\|_{L^p}^{p-1}\\ &\le C\cdot 2^{j(1+\frac{2}{p})}\|\triangle_j u\|_{L^p}\|\triangle_j\theta\|_{L^p}^{p}\\ &\le C\cdot 2^{2j\alpha}\cdot 2^{-j(1+\frac{2}{p}-2\alpha)}\cdot a_j\|\triangle_j\theta\|_{L^p}^{p-1}\|\theta\|^2_{B^{1+\frac{2}{p}-2\alpha}_{p,q}}.\end{aligned}$$

In the same way, we get the estimate for the second term of (1). The third term, denoted by III, is given by

$$\begin{aligned} III &= |\sum_{l\geq j-N}\sum_{|l-m|\leq 1}\int \triangle_j(\triangle_l u\cdot\triangle_m\nabla\theta)\triangle_j\theta|\triangle_j\theta|^{p-2}| \\ &\leq C\cdot\sum_{l\geq j-N}||\triangle_l u||_{L^p}||\triangle_l\theta||_{L^p}||\triangle_j\theta||_{L^p}^{p-2}||\nabla\triangle_j\theta||_{L^\infty} \\ &\leq C\cdot 2^{j(1+\frac{2}{p})}\cdot||\triangle_j\theta||_{L^p}^{p-1}\sum_{l\geq j-N}||\triangle_l\theta||_{L^p}^2 \\ &\leq C\cdot 2^{j(1+\frac{2}{p})} \\ &\quad\cdot||\triangle_j\theta||_{L^p}^{p-1}\sum_{l\geq j-N}2^{-2l(1+\frac{2+2}{p}-2\alpha)} \\ &\quad\cdot 2^{l(1+\frac{2}{p}-2\alpha)}||\triangle_l\theta||_{L^p}\cdot 2^{l(1+\frac{2}{p}-2\alpha)}||\triangle_l\theta||_{L^p} \\ &\leq C\cdot 2^{2j\alpha}\cdot 2^{-j(1+\frac{2}{p}-2\alpha)}\cdot a_j||\triangle_j\theta||_{L^p}^{p-1}||\theta||^2_{B_{p,q}^{1+\frac{2}{p}-2\alpha}}. \end{aligned}$$

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