# Global Well-posedness of the Free Boundary Value Problem of the Incompressible Navier-Stokes Equations With Surface Tension

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**Abstract:** In this paper, we study the global well-posedness of the Navier-Stokes equations with free boundary under the surface tension and gravity in three dimensions. For simplicity, we take a moving domain of finite depth, bounded above by free surface and bounded below by a solid flat bottom. We show that there is a unique, global-in-time solution provided that the initial velocity and the initial profile of the surface are sufficiently small in Sobolev spaces. The main result of this paper is the continuity of the solution at $t = 0$, with initial data of lower regularities. In Appendix, we present local well-posedness results to the problem without surface tension.

## 1. INTRODUCTION

This paper is concerned with the incompressible Navier-Stokes equations with free boundary:

$$(NSF)\begin{cases} v_t + v\cdot\nabla v - \mu\triangle v + \nabla p = 0 \quad \text{in} \quad \Omega_t \\ \nabla\cdot v = 0 \quad \text{in} \quad \Omega_t \\ v = 0 \quad \text{on} \quad S_B \\ \eta_t = v_3 - v_1\partial_x\eta - v_2\partial_y\eta \quad \text{on} \quad S_F \\ pn_i = \mu(v_{i,j} + v_{j,i})n_j + (g\eta - \beta\nabla\cdot(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}))n_i \quad \text{on} \quad S_F \end{cases}$$

where $\Omega_t = \{(x,y,z) : -b < z < \eta(x,y,t)\}$ with two boundaries $S_F = \{(x,y,z) : z = \eta(x,y,t)\}$ and $S_B = \{(x,y,z) : z = -b\}$. Here, $b$ is a constant, $n = (n_1,n_2,n_3)$ is the outward normal vector on $S_F$. $\mu$ is the viscosity, $g$ is the gravitational constant, and $\beta$ is the constant of surface tension. We normalize all the constants by 1. (We follow the Einstein convention where we sum upon repeated indices. Subscripts after commas denote derivatives.) The above system of equations describes the evolution of the free surface and the velocity field defined in $\Omega_t$. We specify the initial compatibility conditions on the initial velocity field $v_0$.

$$\begin{cases} \{(v_0)_{i,j} + (v_0)_{j,i}\}_{\tan} = 0 \quad \text{on} \quad S_F \\ \nabla\cdot v_0 = 0 \quad \text{in} \quad \Omega_0 \\ v_0 = 0 \quad \text{on} \quad S_B \end{cases}$$

where (tan) means the tangential component. The first condition comes from the pressure on the boundary. The boundary condition at the bottom is that the boundary is impenetrable: $v = 0$ which is the boundary condition of the Navier Stokes equations on a fixed domain. This is crucial in order to obtain global-in-time results. We can apply Poincare inequality to control lower order terms by using higher order terms. On the free surface $S_F = \{(x,y,z); z = \eta(x,y,t)\}$, we have three boundary conditions.

• The Kinematic Condition: We represent the free boundary by $d(x,y,z,t) = z - \eta(x,y,t) = 0$. Since $(\partial_t + v \cdot \nabla)$ is tangential to the boundary, $(\partial_t + v \cdot \nabla)(z - \eta(x,y,t)) = 0$. Therefore, $\eta_t = v_3 - v_1 \partial_x \eta - v_2 \partial_y \eta$.

• The Shear Stress Boundary Condition: $\hat{t} \cdot T \cdot \hat{n} = \frac{1}{2}(\hat{t} \cdot \nabla v \cdot \hat{n} + \hat{n} \cdot \nabla v \cdot \hat{t}) = 0$, where $\hat{t}$ is any tangential vector on the boundary and $\hat{n}$ is the outgoing unit normal vector on the boundary given by $\hat{n} = \frac{1}{\sqrt{1+|\nabla\eta|^2}}(-\partial_x \eta, -\partial_y \eta, 1)$.

• The Normal Force Balance: $pn_i = (v_{i,j} + v_{j,i})n_j + \eta n_i - \nabla \cdot (\frac{\nabla \eta}{\sqrt{1+|\nabla\eta|^2}})n_i$

As a starting point, we usually assume that the flow is irrotational in the case of the Euler equations. The fluid motion is described by a velocity potential which is harmonic. We reduce the system into a system where all the functions are evaluated at free surface. But, in the case of the Navier-Stokes equations, it is impossible to assume that the flow is irroatational. The shear stress condition implies that the tangential part of the vorticity on the boundary satisfies

$$w_T = w - (w \cdot \hat{n})\hat{n} = -2\hat{n} \times \nabla v \cdot \hat{n} = -2(\hat{n} \times \nabla) \cdot (\hat{n} \cdot v) + 2u_j((\hat{n} \times \nabla)n_j)$$

where $\hat{n} \times \nabla = (n_2\partial_z - n_3\partial_y, n_3\partial_x - n_1\partial_z, n_1\partial_y - n_2\partial_x)$ is the tangential derivative. This means that vorticity develops at the free surface whenever there is relative flow along a curved surface. This condition prevents a viscous flow from being ir-rotational. The normal part is related to the curvature of the surface. It is evident in two dimensional flow. In a local coordinate system, $w = \hat{n} \cdot \nabla v \cdot \hat{t} - \hat{t} \cdot \nabla v \cdot \hat{n}$. So, from the shear stress condition, we may rewrite $w$ as

$$w = -2\hat{t} \cdot \nabla v \cdot \hat{n} = -2\frac{\partial v}{\partial s} \cdot \hat{v} = -2\frac{\partial}{\partial s}(v \cdot \hat{n}) + 2u \cdot \frac{\partial \hat{n}}{\partial s} = -2\frac{\partial}{\partial s}(v \cdot \hat{n}) + 2(v \cdot \hat{t})\kappa$$

where $\kappa$ is the curvature of the surface. See [9].

There are several papers dealing with the free boundary problem with surface tension. In [3], Beale studied the motion of a viscous incompressible fluid contained in a three dimensional ocean of infinite extent, bounded below by a solid floor and above by an atmosphere of constant pressure. He established the global existence and regularity theorem by taking into account surface tension. His approach is to transform the problem to the equilibrium domain in a way depending on the unknown $\eta$. The entire probelm can be solved by iteration in $K^r$. (For the definition of parabolic-type Sobolev spaces $K^r$, see [3].) In a bounded domain, Coutand-Shkoller [5] used energy methods to establish a priori estimate which allows to find a unique weak solution to the linearized problem in the Lagrangian coordinates. Then, they proved regularity of the weak solution and established the a priori estimates from which they applied the topological fixed point theorem. With additional regularity of $v_t$, uniqueness followed.

When there is no surface tension, Beale [2] answered to two questions. First, he wrote the problem in the Lagrangian formulation so that the domain of the unknowns is fixed in time. He showed the local well-posedness for arbitrary initial data with certain regularity assumptions. The second issue is that for any fixed time interval, solutions exist provided the initial data are sufficiently close to equilibrium. In this case the domain is transformed to the equilibrium domain with flat boundary. Estimates are obtained for the linear problem in which the effect of the change of variables is ignored. The correction terms are then estimated. The contraction mapping theorem is used to obtain the solution. Along the same lines as [3], Lynn-Sylvester [8] showed that viscosity alone will prevent the formation of singularities so that solutions exist globally in time, even without surface tension. Since we are working with the same context of Beale's paper, we present his result here. [3]

**THEOREM 1:** Suppose $r$ is chosen with $3 < r < \frac{7}{2}$. There exists $\delta > 0$ such that for $v_0$ and $\eta_0$ satisfying $|\eta_0|_{H^r(R^2)} + |v_0|_{H^{r-\frac{1}{2}}(\Omega_0)} \leq \delta$ and the compatibility conditions, the problem has a solution $v$, $\eta$ and $p$, where $\eta \in \tilde{K}^{r+\frac{1}{2}}(R^2 \times R^+)$ and $v$ and $p$ are restrictions to the fluid domain $\Omega_t$ of functions defined on $R^3 \times R^+$, with $v \in K^r(R^3 \times R^+)$ and $\nabla p \in K^{r-2}(R^3 \times R^+)$.

**Time regularity of Beale's solution:** Let $r = 3 + \delta$. $v \in K^r$ implies that $v \in H_t^{\frac{s}{2}} H_x^{r-s}$. This is embedded in $C_t H_x^{r-s}$ if $s > 1$. Set $s = 1 + \epsilon$. Then, $r - s = 2 + \delta - \epsilon$. But, the initial data is in $H^{r-\frac{1}{2}}$ and $r - \frac{1}{2} > 2 + \delta - \epsilon$. Therefore, the solution is not continuous in time with values in $H^{r-\frac{1}{2}}$. Alternatively, we can use Proposition 3.1 to show this discrepancy of the regularity of the velocity field. $v \in L_t^2 H_x^r$ and $v_t \in L_t^2 H_x^{r-2}$ implies that $v \in C_t H_x^{r-1}$, but the initial data is in $H^{r-\frac{1}{2}}$ so we lose half a derivative.

In this paper, we establish a priori estimate without transformation of the domain to a fixed domain. Since the moving domain is not translation invariant in the spatial variables, we cannot take usual derivatives to the equation in order to obtain bounds of higher regularities. Instead, we act a second order operator differential $A$ to the equation which is derived from the new expression of the equation. By taking the projection $\mathbb{P}$ onto the divergence free vector field,

$$\mathbb{P}(D_t v) + Av + \nabla \mathscr{H}(n \cdot T \cdot n) + \nabla \mathscr{H}(\eta - \nabla \cdot (\frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}})) = 0$$

where $Aw = -\mathbb{P}\triangle w + \nabla \mathscr{H}(n \cdot T_v \cdot n)$. We will act the operator $A$ to obtain bounds of higher derivatives of $v$. The new equation is a sum of linear and derivative quadratic nonlinear parts:

$$v_t + Av + \nabla \mathscr{H}(\eta - \triangle_0 \eta) = -\mathbb{P}\nabla \cdot (v \otimes v) + \nabla \mathscr{H}(\frac{\nabla \eta |\nabla \eta|^2}{\sqrt{1 + |\nabla \eta|^2}(1 + \sqrt{1 + |\nabla \eta|^2})})$$

From this equation, regularities of $\eta$ are determined by the pre-assigned regularity of the velocity field. Regularity of the velocity field is determined by the second order differential operator $A$. We can do the integration by parts because the nonlinear terms are derivatives. We define a norm of $v, \eta$ as

$$||(v, \eta)||_X = |v|_{L_t^\infty H_x^2} + |\nabla v|_{L_t^2 H_x^2} + |\eta|_{L_t^\infty H_x^3} + |\nabla \mathscr{H}(\eta - F(\eta))|_{L_t^2 H_x^1} + |v_t|_{L_t^2 H_x^1}$$

where $F(\eta) = \nabla \cdot (\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}})$. Since the nonlinear terms are quadratic, we have the following estimate in $\Omega_t$: $||(v,\eta)||_X \lesssim |v_0|_{H^2} + |\eta_0|_{H^3} + ||(v,\eta)||_X^2$. Unfortunately, we cannot perform the iteration directly in $\Omega_t$ because the domain is moving. Traditionally, we fix the domain by the Lagrangian map. We solve the following ODE to obtain the Lagrangian map : $\frac{dx}{dt} = v(t,x), \quad x(0) = y$. This is based on $L^1_t$ estimate of the velocity. But, we have $L^\infty_t$ or $L^2_t$ estimates of the velocity if we do the energy estimates. So, we only expect local in time results. In order to fix the domain, we will make use of Beale's method [3]. We transform the physical domain to the equilibrium domain. The transformed system of equations is given as

$$(LNSF)\begin{cases} w_t - \triangle w + \nabla q = f \quad \text{in} \quad \Omega \\ \nabla \cdot w = 0 \quad \text{in} \quad \Omega \\ w_{i,3} + w_{3,i} = g_i \quad \text{on} \quad \{z=0\} \\ q = w_{3,3} + \int_0^t w_3(s)ds - \triangle_0 \int_0^t w_3(s)ds + g_3 = w_{3,3} + \eta - \triangle_0\eta + g_3 \quad \text{on} \quad \{z=0\} \\ w = w_0 \quad \text{at} \quad t=0, \quad \eta_t = w_3 \quad \text{on} \quad \{z=0\}, \quad w=0 \quad \text{on} \quad \{z=-1\} \end{cases}$$

where $\Omega = \{(x,y,z) : -1 < z < 0\}$ and $(f,g)$ are quadratic nonlinear terms. We will prove that the above system has a unique weak solution in $L^2$. Then, we will establish the regularity of the weak solution by taking tangential derivatives to the equations under more regular initial data and external forces. Regularities of the solution are well matched with the regularities in the a priori estimate in $\Omega_t$. By writing the nonlinear terms explicitly, we can apply the contraction mapping theorem to the problem. The main difference from [2],[3] is that we obtain $L^\infty_t$ bounds on the velocity field and the surface profile so that we do not need to use parabolic-type Sobolev spaces $K^r$ in our estimates. Continuity in time follows using compactness argument.

**THEOREM 2:** Suppose $v_0 \in H^2$ and $\eta_0 \in H^3$. If initial data are sufficiently small, then there is a unique, global-in-time solution $v$, $\eta$, and the pressure $p$ such that

$$|v|_{C_tH^2_x} + |\nabla v|_{L^2_tH^2_x} + |\eta|_{L^\infty_tH^3_x} + |\nabla \mathscr{H}(\eta - F(\eta))|_{L^2_tH^1_x} + |\partial_t v|_{L^2_tH^1_x} + |\nabla p|_{L^2_tH^1_x} \lesssim \epsilon$$

where $\epsilon = |v_0|_{H^2} + |\eta_0|_{H^3}$.

**Remark:** From the energy bound of the boundary, it seems that $\eta$ satisfies that

$$\eta_t = -|\xi|\eta + v \cdot \nabla\eta \cdots$$

But we do not know that this is a good approximate equation to the boundary profile.

This paper is organized as follows. In Chapter 2 of the paper, we establish the a priori estimate in the moving domain by using the energy method. In Chapter 3, we prove the existence and uniqueness by iteration argument. In Chapter 4, we solve the transformed system of equations. In Chapter 5, we give the proof of Proposition 2.2 with some Lemmas. In Chapter 6, we prove

Proposition 2.3, Proposition 2.4, and Proposition 2.5. We need two lemmas to estimate $v_t$. In Chapter 7, we present the change of variables which is used in Chapter 3. In Appendix, we prove the local well-posedness for the incompressible Naiver Stokes equations without surface tension.

**Notations:** We follow the Einstein convention where we sum upon repeated indices. Subscripts after commas denote derivatives.

- $(T_w)$ is a deformation matrix such that $(T_w)_{ij} = \frac{1}{2}(w_{i,j} + w_{j,i})$.
- $< u, u >= \frac{1}{2}\int_\Omega (u_{i,j} + u_{j,i})(u_{i,j} + u_{j,i})dV$
- $\mathscr{H}$ is the harmonic extension operator which extends functions defined on $S_F$ to $\dot{H}^1$ harmonic functions on $\Omega$ with zero Neumann boundary condition on $S_B$. Sometimes we denote by $\bar{f}$.
- $A \lesssim B$ means there is a constant $C$ which only depends on $|\eta_0|_{H^3}$ such that $A \leq CB$.
- $A \lesssim B + \frac{1}{2}D$ means there is a constant $C'$ such that $A \leq C'B + \frac{1}{2}D$. These inequalities come from Young's inequality.
- $R$ is the restriction operator onto a surface.
- $\nabla_0$ means a tangential derivative $\nabla_0 = \nabla_{x,y}$. Similarly, $\triangle_0 = \triangle_{x,y}$.
- $\epsilon > 0$ represents the size of initial data.

## 2. A PRIORI ESTIMATE ON THE MOVING DOMAIN

### (1) BASIC ENERGY ESTIMATE

We do energy estimates to the equations in the physical domain. First, we multiply by $v$ and integrate in the spatial variables.

$$\begin{aligned}
0 &= \int_{\Omega_t} \frac{1}{2}\frac{d}{dt}|v|^2 dV + \int_{\Omega_t} \frac{1}{2}\nabla\cdot(v|v|^2)dV + \int_{\Omega_t}(-\triangle v)\cdot v dV + \int_{\Omega_t}\nabla p\cdot v dV \\
&= \frac{1}{2}\frac{d}{dt}\int_{\Omega_t}|v|^2 dV - \frac{1}{2}\int_{\partial\Omega_t}(v\cdot n)|v|^2 dS + \frac{1}{2}\int_{\partial\Omega_t}(v\cdot n)|v|^2 dS \\
&+ \frac{1}{2}\int_{\Omega_t}|v_{i,j}+v_{j,i}|^2 dV - \int_{\partial\Omega_t}(v_{i,j}+v_{j,i})n_j v_i dS + \int_{\partial\Omega_t} p n_i v_i dS
\end{aligned}$$

Collecting terms, we obtain that

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}|v|^2 dV + \frac{1}{2}\int_{\Omega_t}|v_{i,j}+v_{j,i}|^2 dV + \int_{\partial\Omega_t}(v\cdot n)(\eta - \nabla\cdot(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}))dS = 0$$

From the boundary condition $\eta_t = \sqrt{1+|\nabla\eta|^2}(v\cdot n)$, the above equation can be written as

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}|v|^2 dV + \frac{1}{2}\int_{\Omega_t}|v_{i,j}+v_{j,i}|^2 dV + \int_{\partial\Omega_t}\frac{\eta_t\eta}{\sqrt{1+|\nabla\eta|^2}} - \frac{\eta_t}{\sqrt{1+|\nabla\eta|^2}}\nabla\cdot(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}})dS = 0$$

By the change of variables,

$$\int_{\partial\Omega_t}\frac{\eta_t\eta}{\sqrt{1+|\nabla\eta|^2}} - \frac{\eta_t}{\sqrt{1+|\nabla\eta|^2}}\nabla\cdot(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}})dS = \frac{1}{2}\frac{d}{dt}\int_{R^2}|\eta|^2 + (\sqrt{1+|\nabla\eta|^2}-1)dxdy$$

Integrating in time,

$$\frac{1}{2}|v(t)|^2_{L^2} + |\eta(t)|^2_{L^2} + \int_{R^2} (\sqrt{1+|\nabla\eta(t)|^2} - 1)dxdy + \frac{1}{2}\int_0^t \int_{\Omega_s} |v_{i,j} + v_{j,i}|^2 dV ds = \epsilon$$

Korn's inequality (Lemma 5.5 in Chapter 5) implies that

$$|v(t)|^2_{L^2} + |\eta(t)|^2_{L^2} + \int_{R^2} (\sqrt{1+|\nabla\eta(t)|^2} - 1)dxdy + \int_0^t \int_{\Omega_s} |\nabla v|^2 dV ds \lesssim \epsilon$$

Since $(\sqrt{1+|\nabla\eta|^2} - 1) = \frac{|\nabla\eta|^2}{1+\sqrt{1+|\nabla\eta|^2}}$, we need to show that $|\nabla\eta|_{L^\infty_x}$ is uniformed bounded for all time. (This will be done by higher energy estimates.) Under this boundedness, we obtain the basic $L^2$ bound: $|v|^2_{L^\infty_t L^2_x} + |\nabla v|^2_{L^2_t L^2_x} + |\eta|^2_{L^\infty_t L^2_x} + |\nabla\eta|^2_{L^\infty_t L^2_x} \lesssim \epsilon$

**(2) HIGHER ENERGY ESTIMATE**

In this section, we use the vector field decomposition method in order to rewrite the original equation such that the pressure in the fluid body is expressed as the harmonic extension of the pressure on the boundary, not dealing with the original equation directly. Then, we define a second order differential operator which will be acted on the equation. In this way, we can obtain energy bounds of higher derivatives of the velocity field $v$ and the boundary $\eta$. The main difficulty of this problem is handling the commutator and nonlinear terms containing $\eta$.

Any vector field $X$ in $\Omega$ can be written as a sum of a divergence free vector field and a gradient : $X = u + \nabla\phi$. This is called a Hodge decomposition. From the identity

$$\int_\Omega u\cdot\nabla\phi dV + \int_\Omega (\nabla\cdot u)\phi dV = \int_{\partial\Omega} (u\cdot n)\phi$$

we conclude that $u$ is of divergence free and $u\cdot n = 0$ on $S_B$ is $L^2$ orthogonal to $\nabla\phi$ with $\phi = 0$ on $S_F$. We denote $u$ by $\mathbb{P}X$. Here, we list properties of the operator $\mathbb{P}$.

▶ **Lemma 2.1** (1) It is a bounded operator on $H^s$. (2) If $\phi \in H^1$, then $\mathbb{P}(\nabla\phi) = \nabla\mathscr{H}(\pi)$, where $\phi = \pi$ on $S_F$.

**Proof :** Let us start with the first statement. For any vector field $X$, $X$ can be written as $X = u + \nabla\phi$. Then, $\phi$ satisfies that $\triangle\phi = \nabla\cdot X$ in $\Omega$, $\phi = 0$ on $S_F$ and $n\cdot\nabla\phi = X\cdot n$ on $S_B$. By the usual elliptic theory, $|\nabla\phi|_{H^s} \lesssim |\nabla\cdot X|_{H^{s-1}} + |X\cdot n|_{H^{s-\frac{1}{2}}(S_B)} \lesssim |X|_{H^s}$. Therefore, $|u|_{H^s} = |\mathbb{P}(X)|_{H^s} \lesssim |X|_{H^s}$. We can prove the second property by using the same argument used to the first one. ■

The velocity field $v$ and its time derivative $v_t$ of the problem is in the range of $\mathbb{P}$. Since $p = (v_{i,j} + v_{j,i})n_i n_j + \eta - \nabla\cdot(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}})$ does not vanish on $S_F$, $\mathbb{P}(\nabla p) \neq 0$. By Lemma 2.1, we have the following expression of the pressure: $\mathbb{P}(\nabla p) = \nabla\mathscr{H}(n\cdot T_v\cdot n) + \nabla\mathscr{H}(\eta - \nabla\cdot(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}))$.

We take $\mathbb{P}$ to the equation.

$$\mathbb{P}(D_t v) + Av + \nabla\mathscr{H}(\eta - \nabla\cdot(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}})) = 0 \tag{1}$$

where $Aw = -\mathbb{P}\triangle w + \nabla\mathscr{H}(n\cdot T_v\cdot n)$. Furthermore, by taking the divergence to the original equation, we know that $\nabla p_{v,v} = (I-\mathbb{P})\nabla p$ satisfies the elliptic equation:

$$\begin{cases} -\triangle p_{v,v} = \partial_j v_i \partial_i v_j \quad \text{in} \quad \Omega_t \\ p = 0 \quad \text{on} \quad S_F \\ \nabla p\cdot n = -(\triangle v)\cdot n \quad \text{on} \quad S_B \end{cases}$$

The last boundary condition is obtained by taking the inner product to the equation with the normal vector at the bottom. If we do not take $\mathbb{P}$ to the equation, the original equation can be written as

$$v\cdot\nabla v - \triangle v = -v_t - \nabla\mathscr{H}(\eta - \nabla\cdot(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}})) - \nabla p_{v,v} \tag{2}$$

Thus, this is a Hodge decomposition of the left-hand side $(v\cdot\nabla v - \triangle v)$.

► **Regularity of $v$ and $\eta$:** We investigate the pressure terms to obtain regularities of $\eta$. First, we obtain the regularity of the velocity. If we take $\partial^s$ the usual Navier Stokes equations on $R^3$, multiply by $\partial^s v$ to the equation and integrate in the spatial variables, we have the term $(\partial^s\nabla\cdot(v\otimes v), \partial^s v)$. By the integration by parts, $(\partial^s\nabla\cdot(v\otimes v), \partial^s v) = -(\partial^s(v\otimes v), \partial^{s+1}v)$. The second term is controlled by the Laplacian. We have the product of the velocity fields in the first term. To be an algebra in $H^s$, $s\in\mathbb{N}$, we take $s=2$. As we apply a second order differential operator $A$ to the equation, and do the energy estimates, we have a priori estimate of the velocity field $v$ in $L_t^\infty H^2 \cap L_t^2 H^3$. It requires that $\nabla p \in L_t^2 H_x^1$. Assume that $\eta \in L_t^\infty H_x^a \cap L_t^2 H_x^b$. We have two harmonic functions solving the following elliptic equations. First,

$$(E1)\begin{cases} -\triangle p_1 = 0 \quad \text{in} \quad \Omega_t \\ p_1 = (v_{i,j}+v_{j,i})n_i n_j \quad \text{on} \quad S_F \\ n\cdot\nabla p_1 = 0 \quad \text{on} \quad S_B \end{cases}$$

Since $\nabla v \in L_t^2 H_x^{\frac{3}{2}}$ on $S_F$, $\nabla\eta$ at least should be in $L_t^\infty H_x^{\frac{3}{2}}$. This implies that $a \geq \frac{5}{2}$. As we will see later, $\eta \in L_t^\infty H_x^{\frac{5}{2}}$ is not enough to apply the contraction mapping lemma even though this regularity is well matched with the regularity obtained by solving the transport equation of $\eta$. We will gain regularities of $\eta$ in $L_t^\infty$ by surface tension. Secondly,

$$(E2)\begin{cases} -\triangle p_2 = 0 \quad \text{in} \quad \Omega_t \\ p_2 = \eta - F(\eta) \quad \text{on} \quad S_F \\ n\cdot\nabla p_2 = 0 \quad \text{on} \quad S_B \end{cases}$$

Since $\nabla p_2 \in L_t^2 H_x^1$, it requires that $t = \frac{7}{2}$. By these two elliptic equations, we conclude that we need to obtain $\eta \in L_t^\infty H_x^{\frac{5}{2}+} \cap L_t^2 H_x^{\frac{7}{2}}$. From the equation, we deduce that $\eta_t \in L_t^2 H_x^{\frac{5}{2}}$ and $\eta \in L_t^2 H_x^{\frac{7}{2}}$ (locally in time). This implies that $\eta \in L_t^\infty H_x^3$. These higher regularities of $\eta$ are introduced by surface tension. Now, we rewrite this equation as a sum of linear and nonlinear terms.

$$\begin{aligned} v_t + Av + \nabla \mathscr{H}(\eta - \triangle_0 \eta) &= -\mathbb{P}(v \cdot \nabla v) + \nabla \mathscr{H}(-\triangle_0 \eta + \nabla \cdot (\frac{\nabla \eta}{\sqrt{1+|\nabla \eta|^2}})) \\ &= -\mathbb{P}\nabla \cdot (v \otimes v) + \nabla \mathscr{H}(\frac{\nabla \eta |\nabla \eta|^2}{\sqrt{1+|\nabla \eta|^2}(1+\sqrt{1+|\nabla \eta|^2})}) \end{aligned}$$

Therefore, the right-hand side is derivative quadratic nonlinear terms which are good for the integration by parts. By the regularities obtained just before, the right-hand side is in $\nabla(L_t^2 H_x^2)$. Conversely, if the right-hand side is in $\nabla(L_t^2 H_x^2)$, then we can take two derivatives to the equation. By acting the second order differential operator $A$ to the equation, we establish exactly the same regularities mentioned above. In this way, we can make the argument close.

We go back to (1). We cannot take the usual partial derivatives to the equation because the above equation is not translation-invariant under the influence of the moving boundary. Since we have to perform the integration by parts to generate non-negative quantities, operators acting on the equation should reflect the boundary condition. When we deal with the heat equation on a fixed domain, we can take $\partial_t$ to the equation because the equation is translation invariant in time. From the equation, $\triangle$ has the same effect of $\partial_t$ so that we apply $\triangle$ to the equation in order to obtain bounds of higher derivatives. In our problem, the material derivative $D_t = \partial_t + v \cdot \nabla$ corresponds to $A$ so that we act $A$ to the equation. The second order differential operator $A$ satisfies a nice integration property: For divergence free vector fields $v$ and $w$,

$$\int_{\Omega_t} Av \cdot w dV = \int_{\Omega_t} (-\mathbb{P}\triangle v + \nabla \mathscr{H}(n \cdot T_v \cdot n)) \cdot w dV =< v, w >$$

Since $A$ does not commute with the projection $\mathbb{P}$,

$$A(D_t v) + A(Av) + A(\nabla \mathscr{H}(\eta - \nabla \cdot (\frac{\nabla \eta}{\sqrt{1+|\nabla \eta|^2}}))) = -A\{v \cdot \nabla v - \mathbb{P}(v \cdot \nabla v)\} \tag{3}$$

By commuting $D_t$ with $A$,

$$D_t(Av) + A(Av) + A(\nabla \mathscr{H}(\eta - \nabla \cdot (\frac{\nabla \eta}{\sqrt{1+|\nabla \eta|^2}}))) = [D_t, A]v - A\{v \cdot \nabla v - \mathbb{P}(v \cdot \nabla v)\} \tag{4}$$

where $A(\nabla \mathscr{H}(\eta - \nabla \cdot (\frac{\nabla \eta}{\sqrt{1+|\nabla \eta|^2}}))) = \nabla \mathscr{H}(n \cdot T_{\nabla \mathscr{H}(\eta - \nabla \cdot (\frac{\nabla \eta}{\sqrt{1+|\nabla \eta|^2}}))} \cdot n)$. We multiply by $Av$ to the equation and integrate in the spatial variables over $\Omega_t$.

$$\int_{\Omega_t}[D_t, A]v\cdot AvdV = \frac{1}{2}\frac{d}{dt}|Av|^2_{L^2}+\frac{1}{2}<Av,Av>-\int_{\Omega_t}Av\cdot A\{v\cdot\nabla v-\mathbb{P}(v\cdot\nabla v)\}dV + \int_{\Omega_t}Av\cdot\nabla\mathscr{H}(n\cdot T_{\nabla\mathscr{H}(\eta-\nabla\cdot(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}))}\cdot n)dV$$

Integrating in time,

$$|Av(t)|^2_{L^2}+\int_0^t<Av,Av>ds+\int_0^t\int_{\Omega_s}Av\cdot\nabla\mathscr{H}(n\cdot T_{\nabla\mathscr{H}(\eta-\nabla\cdot(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}))}\cdot n)dVds \lesssim \epsilon+\int_0^t\int_{\Omega_s}[D_t,A]v\cdot AvdVds+\int_0^t\int_{\Omega_s}Av\cdot A\{v\cdot\nabla v-\mathbb{P}(v\cdot\nabla v)\}dVds$$

First of all, we estimate $\int_0^t\int_{\Omega_s}Av\cdot\nabla\mathscr{H}(n\cdot T_{\nabla\mathscr{H}(\eta-\nabla\cdot(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}))}\cdot n)dVds$. We want to single out non-negative terms with cubic or higher order error terms. We define $B$ as $Bv=-\triangle v+\nabla\mathscr{H}(n\cdot T\cdot n)$. We replace $Av$ with $Bv$. Then,

$$\int_{\Omega_t}Av\cdot\nabla\mathscr{H}(n\cdot T_{\nabla\mathscr{H}(\eta-\nabla\cdot(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}))}\cdot n)dV=\int_{\partial\Omega_t}(n\cdot Av)n_in_j\partial_i\partial_j\mathscr{H}(\eta-\nabla\cdot(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}))dS$$
$$=\int_{\partial\Omega_t}(n\cdot Bv)n_in_j\partial_i\partial_j\mathscr{H}(\eta-\nabla\cdot(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}))dS$$
$$=\int_{\partial\Omega_t}\frac{\triangle_0\eta_t}{\sqrt{1+|\nabla\eta|^2}}\triangle_0(\eta-\triangle_0\eta)dS+(\alpha)=\frac{1}{2}\frac{d}{dt}\int_{R^2}(|\triangle_0\eta|^2+|\nabla\triangle_0\eta|^2)dxdy+(\alpha)$$

where $\triangle_0$ means the Laplacian in the horizontal variables. In $(\alpha)$, we have one term at the bottom: $\int_{S_B}(n\cdot Bv)n_in_j\partial_i\partial_j\mathscr{H}(\eta-\nabla\cdot(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}))dS$. Since, at the bottom, $n\cdot Bv=n\cdot\nabla p_{v,v}$ which is small, we will not consider this term in the energy estimate. See (2). From now on, $\partial\Omega_t$ only involves the moving boundary $S_F$. From the above estimate,

$$|Av(t)|^2_{L^2}+\int_0^t<Av,Av>ds+|\triangle_0\eta(t)|^2_{L^2}+|\nabla\triangle_0\eta(t)|^2_{L^2}+\int_0^t(\alpha)ds \lesssim \epsilon+\int_0^t\int_{\Omega_s}[D_t,A]v\cdot AvdVds+\int_0^t\int_{\Omega_s}Av\cdot A\{v\cdot\nabla v-\mathbb{P}(v\cdot\nabla v)\}dVds \tag{5}$$

Secondly, we estimate the last integral in the right-hand side. By Lemma 5.3 and Corollary 5.4 in Chapter 5,

$$\int_{\Omega_t}Av\cdot A\{v\cdot\nabla v-\mathbb{P}(v\cdot\nabla v)\}dV\lesssim|Av|_{L^2}\cdot|A(v\cdot\nabla v-\mathbb{P}(v\cdot\nabla v))|_{L^2}$$
$$\lesssim |Av|^2_{L^2}+\frac{1}{2}|\partial^2(v\cdot\nabla v)|^2_{L^2}\lesssim||v||^4+\frac{1}{2}|\nabla Av|^2_{L^2}$$

where $||v|| = |v|_{L_t^\infty L_x^2} + |Av|_{L_t^\infty L_x^2} + |\nabla v|_{L_t^2 L_x^2} + |\nabla Av|_{L_t^2 L_x^2}$. With the basic energy estimate,

$$\begin{aligned} & |v|^2_{L_t^\infty L_x^2} + |\nabla v|^2_{L_t^2 L_x^2} + |Av|^2_{L_t^\infty L_x^2} + \int < Av, Av > dt + |\eta|^2_{L_t^\infty H_x^3} \\ \lesssim \ & \epsilon + |\int\int_{\Omega_s} Av \cdot ([D_t, A]v) dV dt| + |\int (\alpha) dt| + ||v||^4 + \frac{1}{2}|\nabla Av|^2_{L^2} \end{aligned} \tag{6}$$

We will add $|v_t|^2_{L_t^2 H_x^1}$ and $|\nabla \mathscr{H}(\eta - F(\eta))|^2_{L_t^2 H_x^1}$ to the above energy estimate when we estimate $(\alpha)$ in Chapter 6. The main part of this section is the commutator estimate and the estimate of $(\alpha)$.

▶ **Proposition 2.2. Commutator estimate**

$$\int\int_{\Omega_s} Av \cdot ([D_t, A]v) dV dt \lesssim (|v|_{L_t^\infty L_x^2} + |\nabla v|_{L_t^2 L_x^2} + |Av|_{L_t^\infty L_x^2} + |\nabla Av|_{L_t^2 L_x^2})^3 + \frac{1}{2}|\nabla(Av)|^2_{L_t^2 L_x^2}$$

**Proof :** We assume that the boundary is smooth so that we can apply Sobolev inequalities to obtain the above bound. Then we bound Sobolev constants by $|\eta|_{L_t^\infty H_x^3}$. By taking small initial data $\eta_0$, we can make all Sobolev constants less than a universal number. See Chapter 5. ■

▶ **Proposition 2.3. Estimate of** $(\alpha)$

$$\int (\alpha) dt \lesssim |\eta|^2_{L_t^\infty H_x^3} \cdot ||v||^2 + ||v||^4 + ||v||^2 \cdot |v_t|^2_{L_t^2 H_x^1} + |\eta|^2_{L_t^\infty H_x^3} \cdot |\nabla \mathscr{H}(\eta - F(\eta))|^2_{L_t^2 H_x^1}$$

**Proof :** We will show that $(\alpha)$ consists of cubic or higher order terms. See Chapter 6. ■

▶ **Proposition 2.4. Estimate of** $|v_t|^2_{L_t^2 H_x^1} + |\nabla \mathscr{H}(\eta - F(\eta))|^2_{L_t^2 H_x^1}$

$$\begin{aligned} |v_t|^2_{L_t^2 H_x^1} + |\nabla \mathscr{H}(\eta - F(\eta))|^2_{L_t^2 H_x^1} \ & \lesssim \ \epsilon + ||v||^4 + ||v||^2 \cdot |\eta|^2_{L_t^\infty H_x^3} \\ & + \ ||v||^2 \cdot |v_t|^2_{L_t^2 H_x^1} + |\eta|^2_{L_t^\infty H_x^3} \cdot |\nabla \mathscr{H}(\eta - F(\eta))|^2_{L_t^2 H_x^1} \end{aligned}$$

**Proof :** Since $v_t = 0$ at the bottom, we can use Korn's inequality. To obtain bounds of the gradient of $v_t$, we take $D_t$ to the equation and we do the energy estimate. As before, we need to estimate the commutator term. But, we already know how to obtain the commutator estimate in Proposition 2.2. We control $(\nabla \mathscr{H}(\eta - F(\eta)))$ by using the equation, Proposition 2.2, Proposition 2.3 and $v_t$. See Chapter 6. ■

▶ **Proposition 2.5. Korn-type inequality**

$$|\nabla Av|^2_{L_t^2 L_x^2} \lesssim \int < Av, Av > dt + ||v||^4 + \frac{1}{2}|\nabla Av|^2_{L_t^2 L_x^2} + |\nabla \mathscr{H}(\eta - F(\eta))|^2_{L_t^2 H_x^1}$$

**Proof :** Since $Av$ does not vanish at the bottom, we cannot apply Korn's inequality directly to $Av$. But, we can use the equation to apply Korn's inequality. See Chapter 6. ■

Now, we make the energy bound which is mentioned in Theorem 2. By Proposition 2.5, we replace $(\int < Av, Av > dt)$ with $|\nabla Av|^2_{L^2_t L^2_x}$. Then, we have $|\nabla \mathscr{H}(\eta - F(\eta))|^2_{L^2_t H^1_x}$ in the right-hand side. By Proposition 2.4,

$$\begin{aligned} & |v_t|^2_{L^2_t H^1_x} + |\nabla \mathscr{H}(\eta - F(\eta))|^2_{L^2_t H^1_x} + ||v||^2 + |\eta|^2_{L^\infty_t H^3_x} \\ \lesssim \;& \epsilon + (||v||^2 + |\eta|^2_{L^\infty_t H^3_x} + |v_t|^2_{L^2_t H^1_x} + |\nabla \mathscr{H}(\eta - F(\eta))|^2_{L^2_t H^1_x})^2 \end{aligned}$$

where $\epsilon = |v_0|^2_{H^2} + |\eta_0|^2_{H^3}$. By Lemma 5.3 and Corollary 5.4 in Section 5,

$$\begin{aligned} & |v|^2_{L^\infty_t H^2_x} + |\nabla v|^2_{L^2_t H^2_x} + |\eta|^2_{L^\infty_t H^3_x} + |\nabla \mathscr{H}(\eta - F(\eta))|^2_{L^2_t H^1_x} + |v_t|^2_{L^2_t H^1_x} \\ \lesssim \;& \epsilon + (|v|^2_{L^\infty_t H^2_x} + |\nabla v|^2_{L^2_t H^2_x} + |\eta|^2_{L^\infty_t H^3_x} + |\nabla \mathscr{H}(\eta - F(\eta))|^2_{L^2_t H^1_x} + |v_t|^2_{L^2_t H^1_x})^2 \end{aligned}$$

Finally, we solve the pressure. Since we already know the harmonic extension parts, we only need to solve the Lagrangian multiplier $p_{v,v}$. By Lax-Milgram, there is a unique weak solution $p_{v,v} \in L^2_t H^1_0$. By Poincare inequality, $|\nabla p_{v,v}|_{L^2_t H^1_x} \lesssim |\triangle p_{v,v}|_{L^2_t L^2_x} \lesssim |\partial v \partial v|_{L^2_t L^2_x} + |v \partial v|_{L^2_t L^2_x} \lesssim \epsilon$

## 3. ITERATION, EXISTENCE AND UNIQUENESS

We may iterate the system in the following way. First, we rewrite the equation (1) as

$$v^m_t + \mathbb{P}(v^{m-1} \cdot \nabla v^m) + Av^m + \nabla \mathscr{H}(\eta^m - \triangle_0 \eta^m) + \nabla \mathscr{H}\left(\frac{\nabla \eta^m |\nabla \eta^{m-1}|^2}{\sqrt{1 + |\nabla \eta^{m-1}|^2}(1 + \sqrt{1 + |\nabla \eta^{m-1}|^2})}\right) = 0$$

The evolution of the free surface is given by $\eta^m_t = \sqrt{1 + |\nabla \eta^{m-1}|^2}(v^m \cdot n)$. We solve the system of equations on $\Omega^{m-1}_t$ with initial data uniformly in $m \in N$ : $v^m(\cdot, 0) = v_0$, $\eta^m(\cdot, 0) = \eta_0$. We apply estimates of Proposition 2.2 up to Proposition 2.5. to the iterated system. Then, we have the following estimate: $||(v^m, \eta^m)||_X \lesssim |v_0|_{H^2} + |\eta_0|_{H^3} + ||(v^{m-1}, \eta^{m-1})||_X \cdot ||(v^m, \eta^m)||_X$. But we do not know how to solve the system of equation on $\Omega^{m-1}_t$. Moreover, since $(v^{m+1}, \eta^{m+1})$ and $(v^m, \eta^m)$ are defined in different domains, we cannot take difference of two equations to show that they are Cauchy sequence in $|| \cdot ||_X$. In order to fix the domain, we make use of Beale's method: we transform the physical domain onto the equilibrium domain. Since we project the equation onto the divergence free space, we need the divergence free condition to the velocity field. The linearization is given by the change of variables in a way that the divergence free condition is preserved. We define $\theta(t) : \Omega = \{(x, y, z); -1 < z < 0\} \to \{(x, y, z'); -1 < z' < \eta(x, y, t)\}$ by

$$\theta(x, y, z, t) = (x, y, \bar{\eta}(x, y, t) + z(1 - \bar{\eta}(x, y, t)))$$

where $\bar{\eta}$ is the harmonic extension of $\eta$ into the fluid domain. In order that $\theta$ is a diffeomorphism,

$\eta$ should be small for all time. This will be achieved by energy estimates. We define $v$ on $\theta(\Omega)$ by

$$v_i = \frac{\theta_{i,j}}{J} w_j = \alpha_{ij} w_j$$

where $J = 1 - \bar{\eta} + \partial_z \bar{\eta}(1-z)$, $d\theta = (\theta_{i,j})$. Then, $v$ has divergence free in $\theta(\Omega)$ if and only if $w$ has the same property in $\Omega$. We replace the system of equations of $v$ with that of $w$.

$$v_{i,j} = \zeta_{lj} \partial_l (\alpha_{ik} w_k), \quad v_{i,t} = \alpha_{ij} w_{j,t} + \alpha'_{ij} w_j + (\theta^{-1})'_3 \partial_3 (\alpha_{ij} w_j)$$

where $\zeta = (d\theta)^{-1}$ and $'$ denotes derivatives in $t$. Setting $q = p \circ \theta$, Then the other three terms in the Navier Stokes equations are of the form

$$\alpha_{jk} w_k \zeta_{mj} \partial_m (\alpha_{il} w_l) - \zeta_{kj} \partial_k (\zeta_{mj} \partial_m (\alpha_{il} w_l)) + \zeta_{ki} \partial_k q$$

Multiplying by $(\alpha_{ij})^{-1}$, we have the following equation: $w_t - \triangle w + \nabla q = f(\bar{\eta}, v, \nabla q)$. The normal boundary condition becomes

$$q N_i - \{\zeta_{lj} \partial_l (\alpha_{ik} w_k) + \zeta_{mi} \partial_m (\alpha_{jk} w_k)\} N_j = \{\eta - \nabla \cdot (\frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}})\} N_i$$

with $N = n \circ \theta$. Let $T_1 = (1, 0, \partial_x \eta)$, $T_2 = (0, 1, \partial_y \eta)$. Taking the inner product with $T_1$, $T_2$, and $N$, we obtain that $w_{i,3} + w_{3,i} = g_i(\eta, w)$, $\quad q - w_{3,3} = \eta - \triangle_0 \eta + g_3$, where $g_3 = \triangle \eta - F(\eta) + g'_3 = \nabla \cdot (\frac{\nabla \eta |\nabla \eta|^2}{\sqrt{1+|\nabla \eta|^2}(1+\sqrt{1+|\nabla \eta|^2})}) + g'_3$, $g'_3$ is quadratic in $\eta$ and $w$. Finally, the evolution of the boundary is calculated by using the definition of the new velocity field on $\Omega$. It satisfies that $\eta_t = w_3$ and $\eta(0) = 0$. The compatibility conditions of the initial data is given by

$$\begin{cases} \nabla \cdot w_0 = 0 \quad \text{in} \quad \Omega \\ w_0 = 0 \quad \text{on} \quad \{z = -1\} \\ w(0)_{i,3} + w(0)_{3,i} = g_i(0) \quad \text{on} \quad \{z = 0\}, \quad q(0) = w(0)_{3,3} + g_3(0) \quad \text{on} \quad \{z = 0\} \end{cases}$$

In sum, we have the following linearized system of equations:

$$(LNSF) \begin{cases} w_t - \triangle w + \nabla q = f \quad \text{in} \quad \Omega \\ \nabla \cdot w = 0 \quad \text{in} \quad \Omega \\ w_{i,3} + w_{3,i} = g_i \quad \text{on} \quad \{z = 0\} \\ q = w_{3,3} + \int_0^t w_3(s) ds - \triangle_0 \int_0^t w_3(s) ds + g_3 = w_{3,3} + \eta - \triangle_0 \eta + g_3 \quad \text{on} \quad \{z = 0\} \\ w = w_0 \quad \text{at} \quad t = 0, \quad \eta_t = w_3 \quad \text{on} \quad \{z = 0\}, \quad w = 0 \quad \text{on} \quad \{z = -1\} \end{cases}$$

Now, we study this linearized system of equation. The main issue is the solvability of $(LNS)$. We assume that this solvability is proved. (See Chapter 4.) We define a norm of $(w, \eta, p)$ as

$$||(w, \eta, q)|| = |w|_{L_t^\infty H_x^2} + |w|_{L_t^2 H_x^3} + |\nabla \mathscr{H}(\eta - \triangle \eta)|_{L_t^2 H_x^1} + |\eta|_{L_t^\infty H_x^3} + |\nabla q|_{L_t^2 H_x^1}$$

We calculate nonlinear terms. Principal parts are given by

$$f \sim w\nabla^3\bar{\eta} + \nabla^2\bar{\eta}\nabla w + \nabla^2 w\nabla\bar{\eta} + \nabla\bar{\eta}\nabla q + (l.o.t.), \quad g_i \sim \nabla\eta\nabla w + \nabla(\nabla\eta\nabla\eta) + (l.o.t.)$$

We only need to estimate the highest order terms.

$$\begin{aligned}
|\nabla f|_{L^2_t L^2_x} &\lesssim |\partial(w\nabla^3\bar{\eta})|_{L^2_t L^2_x} + |\partial(\nabla^2\bar{\eta}\nabla w)|_{L^2_t L^2_x} + |\partial(\nabla^2 w\nabla\bar{\eta})|_{L^2_t L^2_x} + |\partial(\nabla q\nabla\bar{\eta})|_{L^2_t L^2_x} \\
&\lesssim |\nabla w\nabla^3\bar{\eta}|_{L^2_t L^2_x} + |w\nabla^4\bar{\eta}|_{L^2_t L^2_x} + |\nabla w\nabla^3\bar{\eta}|_{L^2_t L^2_x} + |\nabla^2 w\nabla^2\bar{\eta}|_{L^2_t L^2_x} \\
&+ |\nabla^3 w\nabla\bar{\eta}|_{L^2_t L^2_x} + |\nabla^2 w\nabla^2\bar{\eta}|_{L^2_t L^2_x} + |\nabla^2 q\nabla\bar{\eta}|_{L^2_t L^2_x} + |\nabla q\nabla^2\bar{\eta}|_{L^2_t L^2_x} \\
&\lesssim |\nabla w|_{L^2_t L^\infty_x}|\nabla^3\bar{\eta}|_{L^\infty_t L^2_x} + |w|_{L^\infty_t L^\infty_x}|\nabla^4\bar{\eta}|_{L^2_t L^2_x} + |\nabla^3 w|_{L^2_t L^2_x}|\nabla\bar{\eta}|_{L^\infty_t L^\infty_x} \\
&+ |\nabla^2 w|_{L^\infty_t L^2_x}|\nabla^2\bar{\eta}|_{L^2_t L^\infty_x} + |\nabla^2 q|_{L^2_t L^2_x}|\nabla\bar{\eta}|_{L^\infty_t L^\infty_x} + |\nabla q|_{L^2_t L^2_x}|\nabla^2\bar{\eta}|_{L^\infty_t L^\infty_x}
\end{aligned}$$

By Lemma 6.2 in Chapter 6, we can replace $|\nabla^4\bar{\eta}|_{L^2_t L^2_x}$ and $|\nabla^2\bar{\eta}|_{L^2_t L^\infty_x}$ with $|\nabla\mathscr{H}(\eta - \triangle\eta)|_{L^2_t H^1_x}$. Hence $|f|_{L^2_t H^1_x} \lesssim (||(w,\eta,q)||)^2$. We do the same calculation to $g$.

$$\begin{aligned}
|\partial^{\frac{3}{2}} g|_{L^2_t L^2_x(R^2)} &\lesssim |\partial^{\frac{3}{2}}(\nabla w)\nabla\eta|_{L^2_t L^2_x(R^2)} + |\nabla w\partial^{\frac{3}{2}}(\nabla\eta)|_{L^2_t L^2_x(R^2)} + |\nabla^{\frac{5}{2}}(\nabla\eta\nabla\eta)|_{L^2_t L^2_x(R^2)} \\
&\lesssim |\partial^{\frac{3}{2}}(\nabla w)|_{L^2_t L^2_x(R^2)}|\nabla\eta|_{L^\infty_t L^\infty_x} + |\nabla w|_{L^2_t L^\infty_x(R^2)}|\partial^{\frac{5}{2}}\eta|_{L^\infty_t L^2_x} + |\nabla\eta|_{L^\infty_t L^\infty_x}|\partial^{\frac{7}{2}}\eta|_{L^2_t L^2_x}
\end{aligned}$$

Therefore, $|g|_{L^2_t H^{\frac{3}{2}}_x(R^2)} \lesssim (||(w,\eta,q)||)^2$.

In order to do the iteration, we treat the nonlinear terms as inputs. The first step is to define $(w^1,\eta^1,q^1)$ . Let $\rho(t)$ be a nice cut-off function in time such that $\rho(0) = 1$. $\mu(x,y)$ is a $C^\infty_c$ function on $R^2$. Then $\eta^1$ is defined as $\eta^1 = (\eta_0 \star \mu(x,y))\rho(t)$. We do the same procedure to define the first velocity field $w^1$. Since $w_0$ is defined on the channel $\Omega$, we restrict $w_0$ interior of the domain. We define $\psi$ which is $C^\infty$ function supported in $-\frac{3}{2} < z < -\frac{1}{4}$. Let $\phi(x,y,z)$ be a nice function such that $\psi = 0$ outside of the domain. Let $\lambda(x,y,z)$ be a $C^\infty_c$ function on $R^3$. Then, we define $w^1$ as $w^1 = ((w_0\psi) \star \lambda(x,y,z))\phi(x,y,z)\rho(t)$. Finally, we define the pressure as $q^1 = \mathscr{H}(w^1_{3,3} + \eta^1 - \triangle\eta^1)$. We do the iteration in the following manner:

$$(LNSF^m)\begin{cases}
w^m_t - \triangle w^m + \nabla q^m = f(w^{m-1},\eta^{m-1},q^{m-1}) \ \text{ in } \ \Omega \\
\nabla\cdot w^m = 0 \ \text{ in } \ \Omega \\
w^m_{i,3} + w^m_{3,i} = g_i(w^{m-1},\eta^{m-1}) \ \text{ on } \ \{z=0\} \\
q^m = w^m_{3,3} + \eta^m - \triangle_0\eta^m + g_3(w^{m-1},\eta^{m-1}) \ \text{ on } \ \{z=0\} \\
\eta^m_t = w^m_3 \ \text{ on } \ \{z=0\} \\
w^m = w_0 \ \text{ at } \ t=0, \quad w^m = 0 \ \text{ on } \ \{z=-1\}
\end{cases}$$

Then, we have the following bound: $||(w^m,\eta^m,q^m)|| \lesssim \epsilon + ||(w^{m-1},\eta^{m-1},q^{m-1})||^2$. Therefore, we conclude that $\{||(w^m,\eta^m,q^m)||\}$ are uniformly bounded if the initial data is small enough. By taking difference of two sequences, we show that $\{||(w^m,\eta^m,q^m)||\}$ are Cauchy sequence. Therefore, we obtain a unique, global-in-time solution if initial data is small enough in $H^2$ by the contraction mapping theorem. The dependence on the initial data of the boundary $\eta_0$ occurs when we take the first iteration. Since we need uniform bound of $\eta \in H^3$, we take $\eta_0 \in H^3$. Having solved the linearized problem, we reverse our steps and obtain a solution of the original problem. By the compactness argument, $v$ is continuous in time with value in $H^2$ because Since $H_0^3$ is compact in $H^2$ and $H^2$ is embedded continuously in $H^1$. We can apply the following Proposition 3.1. But, we cannot say that $\eta$ is continuous in time with value in $H^3$ because $\eta$ is defined on the whole space so that we cannot apply the Rellich compactness, which says the compactness locally in $H^k$.

► **Proposition 3.1. Continuity in time:** Let $m$ to be a nonnegative integer. Suppose $v \in L^2(0,T;H_0^{m+2}(\Omega))$, with $v_t \in L^2(0,T;H^m(\Omega))$. Then, $v \in C([0,T];H^{m+1})$ after possibly being redefined on a set of measure zero. Furthermore, we have the estimate

$$\max_{0\le t\le T} |v(t)|_{H^{m+1}} \lesssim |v|_{L_t^2 H_x^{m+2}} + |v_t|_{L_t^2 H_x^m}$$

**Proof :** See Chapter 5 of Evans [7]. It is based on Lions-Aubin Lemma.

**4. SOLVABILITY OF** $(LNS)$

In this Chapter, we study the linearized problem (LNS) defined in $\Omega = \{(x,y,z); -1 < z < 0\}$. We need the divergence free condition to the velocity field. There are two reasons. First, in the weak formulaton, we act a test function $\phi$ to the equation. Since we only know the explicit form of the pressure on the boundary, we want to remove the interior pressure in the weak form of the equation. By the integration by parts,

$$(\nabla p, \phi) = \int_{\partial\Omega} p(\phi \cdot n)dS - (p, \nabla\cdot\phi)$$

So, we define a function space of test functions such that the divergence free condition holds. Secondly, we will take the projection onto the divergence free space to the equation to obtain $L_t^2$ bounds of the boundary. Therefore, the linearization is given by the change of variables in a way that the divergence free condition is preserved. We have the following linearized system of equations:

$$(LNSF)\begin{cases} w_t - \triangle w + \nabla q = f \quad \text{in} \quad \Omega \\ \nabla \cdot w = 0 \quad \text{in} \quad \Omega \\ w_{i,3} + w_{3,i} = g_i \quad \text{on} \quad \{z=0\} \\ q = w_{3,3} + \int_0^t w_3(s)ds - \triangle_0 \int_0^t w_3(s)ds + g_3 = w_{3,3} + \eta - \triangle_0 \eta + g_3 \quad \text{on} \quad \{z=0\} \\ \eta_t = w_3 \quad \text{on} \quad \{z=0\} \\ w = w_0 \quad \text{at} \quad t=0, \quad w=0 \quad \text{on} \quad \{z=-1\} \end{cases}$$

where $(f, g_i)$ comes from the change of variable. In order to do the iteration, we have to prove that the above system of equations is solvable for given initial data and for given external forces

$f$ and $g$. We will prove that there is a weak solution to $(LNS)$. Then, we will show the regularity of weak solutions. In Proposition 6.2, the nonlinear terms are estimated $L^2$ in time. Even though we only have $L_t^\infty$ bounds of $\eta$ in the energy estimates, we have $L_t^2$ bound of the pressure. We can extract $L_t^2$ bounds of the boundary from this pressure bound. In order to do that we reformulate it as

$$w_t - \mathbb{P}\triangle w + \nabla\mathscr{H}(w_{3,3}) + \nabla\mathscr{H}(\eta - \triangle_0\eta) = \mathbb{P}(f - \nabla\mathscr{H}(g_3)) = F \tag{7}$$

**(1) WEAK FORMULATION**

In this section, we obtain a weak solution of $(LNS)$. First, we define a function space where a weak solution is defined. For any fixed time interval $[0, T]$ with $T < \infty$,

$$\mathscr{V}(T) = \{v \in L_t^2 H_x^1 : \nabla\cdot v = 0,\ \ v = 0 \text{ on } S_B,\ \int_0^t v_3 ds \in L_t^\infty L_x^2(R^2),\ \int_0^t \nabla_0 v_3 ds \in L_t^\infty L_x^2(R^2)\}$$

where the divergence free condition is expressed in the distributional form, i.e. $v$ is orthogonal to gradients of test functions which vanish on $S_F$. This space is almost the same space which is used in the context of the Navier-Stokes equations in a fixed domain except that we include the boundary terms. For test functions we will use in weak formulation, we define $\mathscr{V}$ as

$$\mathscr{V} = \{v \in H_x^1 : \nabla\cdot v = 0,\ \ v = 0 \text{ on } S_B,\ \ v_3 \in L^2(R^2),\ \ \nabla_0 v_3 \in L^2(R^2)\}$$

As usual, we define a space for $v_t$ : $\mathscr{V}'(T) = \{v \in L_t^2 H_x^{-1} : \nabla\cdot v = 0, \ \ v = 0 \text{ on } S_B\}$. Before defining a weak solution to the problem, we resolve the following Proposition.

▶ **Proposition 4.1.** $\mathscr{V}$ is separable.

We have two expression of the pressure on $S_F$ in (LNS). When we study a weak solution, we will use the first expression, while we will use the second expression when we show the regularity and we obtain the a priori estimate.

**Definition:** $(w, w_t) \in \mathscr{V}(T) \times L^2(0, T; \mathscr{V}')$ is a weak solution of $(LNS)$ if for all $v \in \mathscr{V}$,

$$\begin{cases} (w_t, v) + < w, v > + \int_{R^2}(\int_0^t w_3 ds)(v_3)dxdy + \int_{R^2}\nabla_0(\int_0^t w_3 ds)\cdot(\nabla_0 v_3)dxdy = (f, v) + (g, v) \\ \nabla\cdot w = 0 \\ w(\cdot, 0) = w_0 \in L^2 \end{cases}$$

**(2) EXISTENCE AND UNIQUENESS**

In this section, we want to resolve the following existence theorem:

For any $w_0 \in L^2$, $f \in L_t^2 L_x^2(\Omega)$, $g \in L_t^2 L_x^2(R^2)$, there exists a weak solution $(w, w_t) \in \mathscr{V}(T) \times \mathscr{V}'(T)$ such that $w(\cdot, 0) = w_0$

The idea of obtaining a weak solution is clear. Since $\mathscr{V}$ is separable, we can use the Galerkin approximation to the equation. Then we solve an ODE in a fixed time interval $[0,T]$. We do the energy estimate to these approximated equations. Since energy bounds are uniform in the indices, we can pass to the limit. By taking a cut-off function in time, we assert that a weak solution achieves the initial data in $L^2$. From the equation, we show that $w_t \in L_t^2 H_x^{-1}$ which implies that $w \in C_t L_x^2$. In this section, the upper index means the third component of a vector field.

▶ **Galerkin Approximation:** Since $\mathscr{V}$ is separable, there exists a basis $\{\phi_k\}$ which are orthogonal in $L^2$. We approximate $w$ by $w_m(t) = \sum_{j=1}^m \lambda_m^j(t)\phi_j$. We want to select the coefficients $\lambda_m^j(t)$ such that $\lambda_m^j(0) = (w_0, \phi_j)$ and

$$\begin{aligned} & (\partial_t w_m, \phi_j) + < w_m, \phi_j > + \int_{R^2} (\int_0^t w_m^3 ds)(\phi_j^3) dxdy + \int_{R^2} \nabla_0 (\int_0^t w_m^3 ds) \cdot (\nabla_0 \phi_j^3) dxdy \\ = \;& (f, \phi_j) + (g, \phi_j) \end{aligned} \tag{8}$$

We define integrals as $E_{mj} =< \phi_m, \phi_j >$, $H_{mj} = \int_{R^2} (\phi_m^3)(\phi_j^3) dx$, $L_{mj} = \int_{R^2} \nabla_0(\phi_m^3) \cdot (\nabla_0 \phi_j^3) dx$ $F_j = (f, \phi_j)$, $G_j = (g, \phi_j)$. Since $(\partial_t w_m, \phi_j) = \partial_t \lambda_m^j$, (14) is reduced to an ODE.

$$\partial_t \lambda_m^j + E_{mj}\lambda_m^j + H_{mj} \int_0^t \lambda_m^j(s) ds + L_{mj} \int_0^t \lambda_m^j(s) ds = F_j + G_j$$

which is subject to the initial data $\lambda_m^j(0) = (w_0, \phi_j)$. By the standard existence theory for ODE, there exists a unique absolutely continuous function $\lambda_m(t) = \{\lambda_m^j : j = 1, 2, \cdots, m\}$.

▶ **Energy estimate:** For $m = 1, 2, \cdots$, we have the following energy estimate

$$\begin{aligned} & |w_m|^2_{L_t^\infty L_x^2} + |\nabla w_m|^2_{L_t^2 L_x^2} + \sup_{0 \le t \le T} \int_{R^2} | \int_0^t w_m^3 ds|^2 dxdy + \sup_{0 \le t \le T} \int_{R^2} |\nabla_0 \int_0^t w_m^3 ds|^2 dxdy \\ \lesssim \;& |w_0|^2_{L_x^2} + |f|^2_{L_t^2 L_x^2} + |g|^2_{L_t^2 L_x^2} \end{aligned}$$

**Proof:** We multiply by $\lambda_m^j(t)$ to (9) and sum for $j = 1, 2, \cdots, m$ to find that

$$\begin{aligned} & (w_m', w_m) + < w_m, w_m > + \int_{R^2} (\int_0^t w_m^3 ds)(w_m^3) dxdy + \int_{R^2} (\nabla_0 \int_0^t w_m^3 ds) \cdot \nabla_0 (w_m^3) dxdy \\ = \;& \frac{1}{2}\frac{d}{dt}|w_m|^2_{L^2} + < w_m, w_m > + \frac{1}{2}\frac{d}{dt}\{\int_{R^2} |(\int_0^t w_m^3 ds)|^2 dxdy\} + \frac{1}{2}\frac{d}{dt}\{\int_{R^2} |(\nabla_0 \int_0^t w_m^3 ds)|^2 dxdy\} \\ = \;& (f, w_m) + (g, w_m) \end{aligned}$$

Integrating in time, with Young's inequality, we finish energy estimate.

▶ **Passing to the limit:** From the energy estimate, we know that $\{w_m\}$ is uniformly bounded in $L_t^\infty L_x^2 \cap L_t^2 H_x^1$. Up to extraction, $\{w_m\}$ converges to $w$ for the weak star topology in $L_t^\infty L_x^2$ and for the weak topology in $L_t^2 H_x^1$. Since $\{w_m\}$ is bounded in $\mathscr{V}(T)$, for the weak star topology in $L^\infty_t H^1_x$, $(\int_0^t w_m^3 ds)$ and $(\nabla_0 \int_0^t w_m^3 ds)$ converge to $\beta$ and $\gamma$, respectively. By the trace theorem, $\beta = \int_0^t w_3 ds$ and $\gamma = \nabla_0 \int_0^t w_3 ds$ in the sense of distributions. We multiply $\psi \in \mathscr{D}(0,T)$ such that $\psi(T) = 0$ to (8) and integrate in time. By the integration by parts in time,

$$\begin{aligned} - & \int_0^T (w_m, \phi_j)\partial_t \psi dt + \int_0^T < w_m, \psi(t)\phi_j > dt + \int_0^T \int_{R^2} (\int_0^t w_m^3 ds)(\psi(t)\phi_j^3) dxdydt \\ + & \int_0^T \int_{R^2} \nabla_0 (\int_0^t w_m^3 ds) \cdot (\psi(t)\nabla_0 \phi_j^3) dxdydt = (w_m(0), \phi_j)\psi(0) + \int_0^T ((f,\phi_j) + (g,\phi_j))dt \end{aligned}$$

Let $m \to \infty$. Since $w_m(0) \to w_0$ in $L^2$,

$$\begin{aligned} - & \int_0^T (w, \phi_j)\partial_t \psi dt + \int_0^T < w, \psi(t)\phi_j > dt + \int_0^T \int_{R^2} (\int_0^t w_3 ds)(\psi(t)\phi_j^3) dxdydt \\ + & \int_0^T \int_{R^2} \nabla_0 (\int_0^t w_3 ds) \cdot (\psi(t)\nabla_0 \phi_j^3) dxdydt = (w_0, \phi_j)\psi(0) + \int_0^T ((f,\phi_j) + (g,\phi_j))dt \end{aligned}$$

This equality holds for a finite linear combination of $\phi_j's$. Therefore, it holds for $v \in \mathscr{V}$.

$$\begin{aligned} - & \int_0^T (w, v)\partial_t \psi dt + \int_0^T < w, v > \psi(t) dt + \int_0^T \int_{R^2} (\int_0^t w_3 ds)(w_3)\psi(t) dxdydt \\ + & \int_0^T \int_{R^2} \nabla_0 (\int_0^t w_3 ds) \cdot (\nabla_0 v_3)\psi(t) dxdydt = (w_0, v)\psi(0) + \int_0^T ((f,v) + (g,v))dt \qquad (9) \end{aligned}$$

Therefore, we find the following equality in the distribution sense on $(0,T)$

$$(w_t, v) + < w, v > + \int_{R^2} (\int_0^t w_3 ds)(v_3) dxdy + \int_{R^2} \nabla_0 (\int_0^t w_3 ds) \cdot (\nabla_0 v_3) dxdy = (f,v) + (g,v) \quad (10)$$

It remains to show that $w(0) = w_0$. We multiply by $\psi(t)$ to (10) and integrate in time. We get

$$\begin{aligned} - & \int_0^T (w, v)\partial_t \psi dt + \int_0^T < w, v > \psi(t) dt + \int_0^T \int_{R^2} (\int_0^t w_3 ds)(w_3)\psi(t) dxdydt \\ + & \int_0^T \int_{R^2} \nabla_0 (\int_0^t w_3 ds) \cdot (\nabla_0 v_3)\psi(t) dxdydt = (w(0), v)\psi(0) + \int_0^T ((f,v) + (g,v))dt \quad (11) \end{aligned}$$

Comparing (9) and (11), we see that $(w_0 - w(0), v)\psi(0) = 0$ for each $v \in \mathscr{V}$. We choose $\psi$ such that $\psi(0) \neq 0$. Then $w(0) = w_0$. This is proof the existence. Weak solutions are not in $L^2(0,T;\mathscr{V})$ because the trace theorem does not hold in this level of the regularity of weak solutions so that we cannot take the difference of two weak solutions in order to show that a weak solution is unique. We can show uniqueness after proving the regularity results in the next section.

**(3) REGULARITY**

In this section, we study the regularity of the weak solution under higher regularities of the initial data and external forces. These regularities are predicted by the a priori estimate in Chapter 2. Since the domain is invariant under the translation in the horizontal direction, we take tangential derivatives to the equation to obtain bounds of tangential derivatives. Other bounds are deduced from the divergence free condition and the equation itself. We will not denote the domain and its boundary when we do the integration. It will be clear from the context. We will not obtain full bounds of $w$ in $L^\infty_t H^2_x$. There are two reasons not to do it. First, by compactness, $w \in L^2_t H^3_0$ and $w_t \in L^2_t H^1$ imply that $w \in C_t H^2$ so that we do not need to estimate $w$ in $L^\infty_t H^2$. Secondly, in fact, we cannot obtain the full derivative by our method because we only assume that $f \in L^2_t H^1_x$ and $g \in L^2_t H^{\frac{3}{2}}_x$. But, in order to obtain all derivatives in $L^\infty_t H^2_x$, we have to take one more time derivative to the equation. Then, we need regularities of $f_t$ and $g_t$. Since we want to treat nonlinear terms as simple as possible, we will not take time derivatives to the equation. Finally, we will use Korn's inequality repeatedly by the bottom boundary condition.

▶ **Proposition 4.2.** Suppose that $(w, w_t) \in \mathscr{V}(T) \times \mathscr{V}'(T)$ is the weak solution such that the initial data satisfies the compatibility condition. Let $w_0 \in H^2$, $f \in L^2_t H^1_x$, $g \in L^2_t H^{\frac{3}{2}}_x$. Then, $w \in L^2_t H^3_x$, $w_t \in L^2_t H^1_x$, $\eta \in L^\infty_t H^3_x$ and $\nabla q \in L^2_t H^1_x$. Moreover, $(w, w_t, \eta, p)$ satisfies the following energy bound: $|w|_{L^\infty_t H^2_x} + |\nabla w|_{L^2_t H^2_x} + |\eta|_{L^\infty_t H^3_x} + |\nabla q|_{L^2_t H^1_x} \lesssim \epsilon + |f|_{L^2_t H^1_x} + |g|_{L^2_t H^{\frac{3}{2}}_x}$, where $\epsilon$ only depends on $|w_0|_{H^2}$, not on $|\eta_0|_{H^3}$ because $\eta_0$ is equal to zero on the equilibrium domain.

**Proof :** Here, we will use the following notations. $A \lesssim B$ means there is a constant $C$ which is a universal constant such that $A \leq CB$. As we will use Young's inequality repeatedly, $A \lesssim B + \frac{1}{2}D$ means there is a constant $C'$ such that $A \leq C'B + \frac{1}{2}D$. First, we obtain the basic energy estimate. We multiply by $w$ to the equation and do the integration by parts.

$$\frac{1}{2}\frac{d}{dt}|w|^2_{L^2} + < w, w > + (q - w_{3,3}, w_3) - (w_i, g_i) = (f, w)$$

From the boundary condition, $(p - w_{3,3}, w_3) = (\eta - \triangle\eta + g_3, w_3) = \frac{1}{2}\frac{d}{dt}(|\eta|^2_{L^2} + |\nabla\eta|^2_{L^2}) + (g_3, w_3)$. By the trace theorem and Young's inequality, $(g_i, w_i) \lesssim |g|^2_{L^2(R^2)} + \frac{1}{2}|w|^2_{L^2} + \frac{1}{2}|\nabla w|^2_{L^2}$. Integrating in time,

$$|w|^2_{L^\infty_t L^2_x} + |\nabla w|^2_{L^2_t L^2_x} + |\eta|^2_{L^\infty_t L^2_x} + |\nabla\eta|^2_{L^\infty_t L^2_x} \lesssim \epsilon + |f|^2_{L^2_t L^2_x} + |g|^2_{L^2_t L^2_x} + \frac{1}{2}|w|^2_{L^2_t L^2_x}$$

Since $w = 0$ on the bottom, we can apply Korn's inequality.

$$|w|^2_{L^\infty_t L^2_x} + |\nabla w|^2_{L^2_t L^2_x} + |\eta|^2_{L^\infty_t L^2_x} + |\nabla\eta|^2_{L^\infty_t L^2_x} \lesssim \epsilon + |f|^2_{L^2_t L^2_x} + |g|^2_{L^2_t L^2_x}$$

We can control all lower order terms by using this energy inequality. Since terms containing $g_1$ and $g_2$ have the same estimates, we assume that $g_1 = g_2 = 0$. Next, we obtain bounds of derivatives. We multiply the equation by $D_{-h}D_h w$ and integrate in the spatial variables. Here $D_h$ means a tangential derivative and $D_{-h} = -D_h$. Since $D_h w = 0$ at the bottom, only boundary terms on

$z = 0$ are involved when we do the integration by parts. By the integration by parts,

$$\frac{1}{2}\frac{d}{dt}|D_h w|^2_{L^2} + < D_h w, D_h w > + (q - w_{3,3}, D_{-h} D_h w_3) \lesssim |f|^2_{L^2}$$

From the boundary condition, $\frac{1}{2}\frac{d}{dt}|D_h w|^2_{L^2} + < D_h w, D_h w > + (\eta - \triangle_0 \eta, D_{-h} D_h w_3) + (D_h g_3, D_h w_3) \lesssim |f|^2_{L^2}$. By the duality argument in $(D_h g_3, D_h w_3)$ and Young's inequality,

$$\begin{aligned} & \frac{d}{dt}|D_h w|^2_{L^2} + < D_h w, D_h w > + \frac{d}{dt}(|D_h \eta|^2_{L^2} + |D_h \nabla_0 \eta|^2_{L^2}) \\ \lesssim\ & |D_h g_3|^2_{\dot{H}^{-\frac{1}{2}}(R^2)} + \frac{1}{2}|D_h w|^2_{\dot{H}^{\frac{1}{2}}(R^2)} + |f|^2_{L^2} \lesssim |g|^2_{H^{\frac{1}{2}}(R^2)} + \frac{1}{2}|D_h w|^2_{H^{\frac{1}{2}}(R^2)} + |f|^2_{L^2} \end{aligned}$$

By the trace theorem,

$$\frac{d}{dt}(|D_h w|^2_{L^2} + |\nabla \eta|^2_{L^2} + |\nabla^2 \eta|^2_{L^2}) + < D_h w, D_h w > \lesssim \frac{1}{2}|D_h \nabla w|^2_{L^2} + |f|^2_{L^2} + |g|^2_{H^{\frac{1}{2}}(R^2)} + |D_h w|^2_{L^2}$$

Integrating in time, with Korn's inequality,

$$|D_h w|^2_{L^\infty_t L^2_x} + |D_h \nabla w|^2_{L^2_t L^2_x} + |\nabla \eta|^2_{L^\infty_t L^2_x} + |\nabla^2 \eta|^2_{L^\infty_t L^2_x} \lesssim \epsilon + |f|^2_{L^2_t L^2_x} + |g|^2_{L^2_t H^{\frac{1}{2}}_x(R^2)}$$

We need to obtain bounds of $|w_{1,33}|_{L^2_t L^2_x}$, $|w_{2,33}|_{L^2_t L^2_x}$ and $|w_{3,33}|_{L^2_t L^2_x}$. Since $\nabla \cdot w = 0$, $w_{3,33} = -w_{1,13} - w_{2,23}$, so that $|w_{3,33}|_{L^2} \leq 2|D_h \nabla w|_{L^2}$. From the equation, $w_{i,33} = -w_{i,jj} + w_{i,t} + \partial_i q$. We can replace $|D_h \nabla w|_{L^2}$ with $|\nabla^2 w|_{L^2}$ with adding $|w_t|_{L^2} + |\partial_i q|$ to the right-hand side. Finally, $\nabla q = w_t - \triangle w + f$ implies that

$$|\partial_3 q|^2_{L^2_t L^2_x} \lesssim |w_{3,t}|^2_{L^2_t L^2_x} + |D_h \nabla w_3|^2_{L^2_t L^2_x} + |f_3|^2_{L^2_t L^2_x} \lesssim |w_t|^2_{L^2_t L^2_x} + |f|^2_{L^2_t L^2_x} + |g|^2_{L^2_t H^{\frac{1}{2}}_x(R^2)} + |D_h w|^2_{L^2_t L^2_x}$$

We have to use a different method to bound $\partial_i q$ because we cannot bound $|\partial_{33} w_i|_{L^2}$ by $|D_h \nabla w|_{L^2}$. As Coutand-Shkoller did in [5], the idea is that we trade $\partial_i$ in $\partial_i q$ with $\partial_3$ in $\partial_{33} w_i$ by the integration by parts twice. We set $u = (\partial_1 q, \partial_2 q, 0)$.

$$\begin{aligned} |u|^2_{L^2} &= (u, u) = (u, f - w_t + \triangle w) = (u, f - w_t) - < u, w > \\ &= (u, f - w_t) - \int ((w_{k,l} + w_{l,k})\partial_{kl} q + \frac{1}{2}(w_{i,3} + w_{3,i})\partial_{i3} q) dV \\ &= (u, f - w_t) + \int ((w_{k,kl} + w_{l,kk})\partial_l q + \frac{1}{2}(w_{i,i3} + w_{3,ii})\partial_3 q) dV \end{aligned}$$

where we do not have boundary integrals because $u_3 = 0$, $g_1 = g_2 = 0$ and $k, l = 1, 2$. By Young's inequality, $\partial_i q$ has the same bound as $\partial_3 q$. Therefore,

$$|D_h w|^2_{L^\infty_t L^2_x} + |\nabla^2 w|^2_{L^2_t L^2_x} + |\nabla \eta|^2_{L^\infty_t H^1_x} + |\nabla q|^2_{L^2_t L^2_x} \lesssim \epsilon + |f|^2_{L^2_t L^2_x} + |g|^2_{L^2_t H^{\frac{1}{2}}_x(R^2)} + |w_t|^2_{L^2_t L^2_x}$$

We take one more derivative. We multiply by $D_{-h}D_hD_{-h}D_hw$ to the equation.

$$(w_t, D_{-h}D_hD_{-h}D_hw) + (-\triangle w, D_{-h}D_hD_{-h}D_hw) + (\nabla p, D_{-h}D_hD_{-h}D_hw) = (f, D_{-h}D_hD_{-h}D_hw)$$

By the integration by parts,

$$\frac{1}{2}\frac{d}{dt}|D_{-h}D_hw|^2_{L^2} + < D_{-h}D_hw, D_{-h}D_hw > +(q - w_{3,3}, D_{-h}D_hD_{-h}D_hw_3) \lesssim |D_hf|^2_{L^2}$$

By the boundary condition, the trace theorem and the duality argument,

$$\frac{d}{dt}(|D_{-h}D_hw|^2_{L^2} + |\nabla^2\eta|^2_{L^2} + |\nabla^3\eta|^2_{L^2}) + |\nabla(D_hD_{-h}w)|^2_{L^2_tL^2_x} \lesssim |\nabla f|^2_{L^2} + |g|^2_{H^{\frac{3}{2}}(R^2)}$$

As before, we can replace $|\nabla(D_hD_{-h}w)|^2_{L^2_tL^2_x}$ with $|\nabla^3w|^2_{L^2_tL^2_x}$ with $|\nabla w_t|^2_{L^2_tL^2_x}$ in the right-hand side. integrating in time,

$$|D_{-h}D_hw|^2_{L^\infty_tL^2_x} + |\nabla^3w|^2_{L^2_tL^2_x} + |\nabla^2\eta|^2_{L^\infty_tH^1_x} + |\nabla^2q|^2_{L^2_tL^2_x} \lesssim \epsilon + |\nabla f|^2_{L^2_tL^2_x} + |g|^2_{L^2_tH^{\frac{3}{2}}_x} + |D_hw_t|^2_{L^2_tL^2_x}$$

Finally, we need to obtain the bound $|w_t|^2_{L^2_tL^2_x} + |D_hw_t|^2_{L^2_tL^2_x}$. First, we multiply by $w_t$ to the equation.

$$(w_t, w_t) + (-\triangle w, w_t) + (\nabla q, w_t) = (f, w_t)$$

By the integration by parts, $|w_t|^2_{L^2} + \frac{1}{2}\frac{d}{dt} < w, w > +(q - w_{3,3}, w_{3,t}) \lesssim |f|^2_{L^2}$. From the boundary condition, $|w_t|^2_{L^2} + \frac{1}{2}\frac{d}{dt} < w, w > +(\eta - \triangle\eta + g_3, w_{3,t}) \lesssim |f|^2_{L^2}$. We estimate boundary terms.

$$(\eta - \triangle\eta, \eta_{tt}) = \frac{d}{dt}(\eta - \triangle\eta, \eta_t) - (\eta_t - \triangle\eta_t, \eta_t) = \frac{d}{dt}(\eta - \triangle\eta, \eta_t) - |\eta_t|^2_{L^2} - |\nabla\eta_t|^2_{L^2}$$

By the trace theorem, $(g_3, w_{3,t}) \lesssim |g_3|^2_{L^2(R^2)} + \frac{1}{2}|w_{3,t}|^2_{L^2(R^2)} \lesssim |g_3|^2_{L^2} + \frac{1}{4}|w_{3,t}|^2_{L^2} + \frac{1}{4}|\nabla w_{3,t}|^2_{L^2}$. Integrating in time, with the Korn's inequality,

$$\begin{aligned}
& |w_t|^2_{L^2_tL^2_x} + |\nabla w|^2_{L^\infty_tL^2_x} \\
\lesssim\ & \epsilon + |f|^2_{L^2_tL^2_x} + |(\eta - \triangle\eta, \eta_t)| + |g|^2_{L^2_tL^2_x(R^2)} + |\eta_t|^2_{L^2_tL^2_x} + |\nabla\eta_t|^2_{L^2_tL^2_x} + |w|^2_{L^\infty_tL^2_x} + \frac{1}{4}|\nabla w_{3,t}|^2_{L^2} \\
\lesssim\ & \epsilon + |f|^2_{L^2_tL^2_x} + \frac{1}{2}|\eta|^2_{L^\infty_tH^2_x} + |\eta_t|^2_{L^\infty_tL^2_x} + |g|^2_{L^2_tH^{\frac{1}{2}}_x} + |\eta_t|^2_{L^2_tH^1_x} + |w|^2_{L^2_tL^2_x} + \frac{1}{4}|\nabla w_{3,t}|^2_{L^2}
\end{aligned}$$

By the trace theorem,

$$|\eta_t|^2_{L^\infty_tL^2_x} = |w_3|^2_{L^\infty_tL^2_x(R^2)} \lesssim |w|^2_{L^\infty_tL^2_x} + \frac{1}{2}|\nabla w|^2_{L^\infty_tL^2_x} \qquad |\eta_t|^2_{L^2_tL^2_x} = |w_3|^2_{L^2_tL^2_x(R^2)} \lesssim |w|^2_{L^2_tL^2_x} + \frac{1}{2}|\nabla w|^2_{L^2_tL^2_x}$$

$$|\nabla\eta_t|^2_{L^2_tL^2_x} = |\nabla w_3|^2_{L^2_tL^2_x(R^2)} \lesssim |w|^2_{L^2_tL^2_x} + \frac{1}{2}|\nabla^2w|^2_{L^2_tL^2_x}$$

Therefore, we have the following bound:

$$\begin{aligned}|w_t|^2_{L^2_tL^2_x} + |\nabla w|^2_{L^\infty_tL^2_x} &\lesssim \epsilon + |f|^2_{L^2_tL^2_x} + \frac{1}{2}|\eta|^2_{L^\infty_tH^2_x} + |g|^2_{L^2_tH^{\frac{1}{2}}_x} + \frac{1}{2}|\nabla w|^2_{L^2_tH^1_x} + |w|^2_{L^2_tL^2_x} + \frac{1}{4}|\nabla w_{3,t}|^2_{L^2} \\ &\lesssim \epsilon + |f|^2_{L^2_tL^2_x} + \frac{1}{2}|\eta|^2_{L^\infty_tH^2_x} + |g|^2_{L^2_tH^{\frac{1}{2}}_x} + \frac{1}{2}|\nabla w|^2_{L^2_tH^1_x} + |w|^2_{L^2_tL^2_x} + \frac{1}{2}|D_hw_t|^2_{L^2}\end{aligned}$$

We take one more derivative. We multiply by $D_{-h}D_hw_t$ to the equation. By integrating in the spatial variables,

$$|D_hw_t|^2_{L^2} + \frac{1}{2}\frac{d}{dt} < D_hw, D_hw > +(q - w_{3,3}, D_{-h}D_hw_{3,t}) = (f, D_{-h}D_hw_t)$$

By the same method used before, we obtain that

$$|D_hw_t|^2_{L^2_tL^2_x} + |D_h\nabla w|^2_{L^\infty_tL^2_x} \lesssim \epsilon + |\nabla f|^2_{L^2_tL^2_x} + \frac{1}{2}|\nabla\eta|^2_{L^\infty_tH^2_x} + |g|^2_{L^2_tH^{\frac{3}{2}}_x} + \frac{1}{2}|\nabla^2 w|^2_{L^2_tH^1_x}$$

In sum,

$$|w|^2_{L^\infty_tH^1_x} + |D_h\nabla w|^2_{L^\infty_tL^2_x} + |\nabla w|^2_{L^2_tH^2_x} + |\eta|^2_{L^\infty_tH^3_x} + |\nabla q|^2_{L^2_tH^1_x} \lesssim \epsilon + |f|^2_{L^2_tH^1_x} + |g|^2_{L^2_tH^{\frac{3}{2}}_x}$$

From the equation, we conclude that

$$|w|^2_{L^\infty_tH^1_x} + |D_h\nabla w|^2_{L^\infty_tL^2_x} + |\nabla w|^2_{L^2_tH^2_x} + |w_t|^2_{L^2_tH^1_x} + |\eta|^2_{L^\infty_tH^3_x} + |\nabla q|^2_{L^2_tH^1_x} \lesssim \epsilon + |f|^2_{L^2_tH^1_x} + |g|^2_{L^2_tH^{\frac{3}{2}}_x}$$

We finish proof of Proposition. ■

**Remark 1:** If we try to obtain the full derivative of $w$ in $L^\infty_tH^2_x$, we need to obtain bounds of $|w_{1,33}|_{L^\infty_tL^2_x}$ and $|w_{2,33}|_{L^\infty_tL^2_x}$. From the equation, $|w_{i,33}|^2_{L^\infty_tL^2_x} \lesssim |w_{i,t}|^2_{L^\infty_tL^2_x} + |f|^2_{L^\infty_tL^2_x} + |\partial_iq|^2_{L^\infty_tL^2_x}$. We estimate $|w_{i,t}|^2_{L^\infty_tL^2_x}$ and $|\partial_iq|^2_{L^\infty_tL^2_x}$.

$$\begin{aligned}|\partial_ip|^2_{L^\infty_tL^2_x} &\lesssim |\nabla p_{v,v}|^2_{L^\infty_tL^2_x} + |\nabla\mathscr{H}(\eta - \triangle\eta)|^2_{L^\infty_tL^2_x} + |\nabla\mathscr{H}(w_{3,3})|^2_{L^\infty_tL^2_x} + |\nabla\mathscr{H}(g)|^2_{L^\infty_tL^2_x} \\ &\lesssim |v\cdot\nabla v|^2_{L^\infty_tL^2_x} + |\eta - \triangle\eta|^2_{L^\infty_tH^{\frac{1}{2}}_x} + |w_{3,3}|^2_{L^\infty_tH^{\frac{1}{2}}_x} + |g|^2_{L^\infty_tH^{\frac{1}{2}}_x} \\ &\lesssim \frac{1}{10}|v|^2_{L^\infty_tL^\infty_x} + |\nabla v|^2_{L^2_tL^2_x} + |\eta|^2_{L^\infty_tH^3_x} + |w|^2_{L^\infty_tL^2_x} + |D_hw|^2_{L^\infty_tL^2_x} + |g|^2_{L^\infty_tH^{\frac{1}{2}}_x}\end{aligned}$$

In order to obtain $|w_{i,t}|^2_{L^\infty_tL^2_x}$, we take $\partial_t$ to the equation, multiply by $w_t$ and do the integration by parts. Then, $|w_{i,t}|^2_{L^\infty_tL^2_x} \lesssim |\nabla^{-1}f_t|^2_{L^2_tL^2_x} + |\nabla^{-\frac{1}{2}}g_t|^2_{L^2_tL^2_x}$. But, it is not clear how we estimate $\nabla^{-1}f_t$ and $\nabla^{-\frac{1}{2}}g_t$.

**Remark 2:** If we have $g_1$ and $g_2$, then, for $k, l, m, i = 1, 2$,

$$\begin{aligned}|u|^2_{L^2} &= (u,u) = (u, f - w_t + \triangle w) = (u, f - w_t) - < u, w > + \int_{R^2} u_m g_m dS \\ &= (u, f - w_t) - \int ((w_{k,l} + w_{l,k})\partial_{kl} q + \frac{1}{2}(w_{i,3} + w_{3,i})\partial_{i3} q) dV + \int_{R^2} \partial_m q g_m dS \\ &= (u, f - w_t) + \int ((w_{k,kl} + w_{l,kk})\partial_l q + \frac{1}{2}(w_{i,i3} + w_{3,ii})\partial_3 q) dV + \int_{R^2} \partial_m q g_m dS\end{aligned}$$

Therefore, $|\partial_i q|^2_{L^2} \lesssim \epsilon + |f|^2_{L^2_t L^2_x} + |g|^2_{L^2_t H^{\frac{1}{2}}_x (R^2)} + |w_t|^2_{L^2_t L^2_x} + \frac{1}{2}|\nabla q|^2_{L^2(R^2)}.$. We can move $\frac{1}{2}|\nabla q|^2_{L^2(R^2)}$ to the left-hand side because we have the factor $\frac{1}{2}$ in front of $|\nabla q|^2_{L^2(R^2)}$. When we take one more derivative to the equation, we have error terms of the form: $\int (\partial^2 q)(\partial g) dxdy$. By duality argument, $\int (\partial^2 q)(\partial g) dxdy \lesssim \frac{1}{2}|\partial^{\frac{3}{2}} q|^2_{L^2(R^2)} + |g|^2_{H^{\frac{3}{2}}(R^2)}$. Combining both, we have $\frac{1}{2}|\nabla q|^2_{H^1(R^2)}$ in the right-hand side. We apply the trace theorem and move it to the left-hand side.

Now, we take the projection $\mathbb{P}$ to the equation: $w_t - \mathbb{P}(\triangle w) + \mathbb{P}(\nabla q) = \mathbb{P}(f)$. Since $w$ and its tangential derivatives are of divergence free and is zero on the bottom boundary, for $k \geq 0$,

$$(-\triangle w, (D_h)^k w) = (-\mathbb{P}\triangle w, (D_h)^k w) - ((I - \mathbb{P})\triangle w, (D_h)^k w) = (-\mathbb{P}\triangle w, (D_h)^k w)$$

$$(\nabla q, (D_h)^k w) = (\mathbb{P}(\nabla q), (D_h)^k w) + ((I - \mathbb{P})\nabla q, (D_h)^k w) = (\mathbb{P}\nabla q, (D_h)^k w)$$

Therefore, we have the same estimates as before for the projected equation. $\nabla \mathscr{H}(\eta - \triangle \eta) = \mathbb{P}\nabla q - \nabla \mathscr{H}(w_{3,3}) - \nabla \mathscr{H}(g_3)$ infers that

$$|\nabla \mathscr{H}(\eta - \triangle \eta)|^2_{L^2_t H^1_x} \leq |\nabla q|^2_{L^2_t H^1_x} + |\nabla \mathscr{H}(w_{3,3})|^2_{L^2_t H^1_x} + |\nabla \mathscr{H}(g_3)|^2_{L^2_t H^1_x}$$

By Proposition 4.2,

$$|\nabla w|_{L^2_t H^2_x} + |\eta|_{L^\infty_t H^3_x} + |\nabla q|_{L^2_t H^1_x} + |\nabla \mathscr{H}(\eta - \triangle \eta)|^2_{L^2_t H^1_x} \lesssim \epsilon + |f|_{L^2_t H^1_x} + |g|_{L^2_t H^{\frac{3}{2}}_x (R^2)}$$

## 5. PROOF OF PROPOSITION 2.2

Before proving Proposition 2.2, we collect lemmas which are required to prove it.

► **Lemma 5.1. Sobolev Inequalities in 3D:** For a domain with regular boundary,

(1) $|f|_{L^4} \leq C|f|^{\frac{1}{4}}_{L^2} \cdot |\nabla f|^{\frac{3}{4}}_{L^2}$ (2) $|f|_{L^\infty} \leq C|f|_{H^2}$

Here, $C$ depends on the regularity of the boundary. Since $\eta \in L^\infty_t H^3_x$, we can use this lemma to Proposition 2.2.

▶ **Lemma 5.2. Trace Theorem:** Let $\Omega$ be a domain in $R^n$ having the uniform $C^m$ regularity property and suppose there exists a simple $(m,p)$ extension operator $E$ for $\Omega$. If $mp < n$ and $p \leq q \leq \frac{(n-1)p}{n-mp}$. Then, $W^{m,p} \to L^q(\partial\Omega)$. If $mp = n$, then the above holds for $p \leq q < \infty$. See [1]. In this section we will use $H^1(\Omega) \to L^2(\partial\Omega)$. In Section 4, we applied a sharp version $H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$. Since $\eta \in L_t^\infty H_x^3$, we have the enough boundary regularity to apply Trace theorem.

▶ **Lemma 5.3. Unique solvability of an elliptic equation, part I**

$$\begin{cases} Av = f \quad \text{in} \quad \Omega \\ \hat{t} \cdot T \cdot \hat{n} = 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

Then, under the divergence free condition, $|v|_{H^r} \lesssim |Av|_{H^{r-2}}$ for $r = 2, 3$. See Lemma 3.3 in Beale [2]. We have the following corollary of Lemma 5.3.

▶ **Corollary 5.4:** Sobolev inequalities involving $Av$.

(1) $|\partial v|_{L^2} \lesssim |v|_{L^2}^{\frac{1}{2}} \cdot |Av|_{L^2}^{\frac{1}{2}}$ (2) $|Av|_{L^2} \lesssim |\partial v|_{L^2}^{\frac{1}{2}} \cdot |\nabla(Av)|_{L^2}^{\frac{1}{2}}$ (3) $|\nabla^3 v|_{L^2} \lesssim |\nabla Av|_{L^2} + |\nabla v|_{L^2}$

**Proof:** The first inequality is derived from Lemma 5.3 and interpolation. That is,

$$|\partial v|_{L^2} \lesssim |v|_{L^2}^{\frac{1}{2}} \cdot |\nabla^2 v|_{L^2}^{\frac{1}{2}} \lesssim |v|_{L^2}^{\frac{1}{2}} \cdot |Av|_{L^2}^{\frac{1}{2}}$$

The second inequality is obtained by using the divergence free condition of $Av$ :

$$|Av|_{L^2} = (Av, Av)^{\frac{1}{2}} =< v, Av >^{\frac{1}{2}} \leq |\partial v|_{L^2}^{\frac{1}{2}} \cdot |\nabla(Av)|_{L^2}^{\frac{1}{2}}$$

The last inequality is obtained by taking $r = 3$ in Lemma 5.3.

**Lemma 5.5. Korn's inequality:** $|v|_{H^1}^2 \leq C < v, v >$. See Lemma 2.7 in [2].

Now, we prove Proposition 2.2. Since $(Av \cdot [D_t, A]v)$ is cubic, we can distribute $L_t^\infty$ and $L_t^2$ as we want. We expect that $\int\int_{\Omega_s} Av \cdot ([D_t, A]v) dV dt$ is of the form of $(LHS)^2 + \frac{1}{2}(LHS)$. Here, $(LHS) = ||v||^2$.

▶ **Proposition 2.2. Commutator estimate**

$$\int\int_{\Omega_s} Av \cdot ([D_t, A]v) dV dt \lesssim (|v|_{L_t^\infty L_x^2} + |\nabla v|_{L_t^2 L_x^2} + |Av|_{L_t^\infty L_x^2} + |\nabla Av|_{L_t^2 L_x^2})^3 + \frac{1}{2}|\nabla(Av)|_{L_t^2 L_x^2}^2$$

**Proof:** We may expect the usual commutator estimate :

$$
\begin{aligned}
(\blacktriangleright) \quad & \int_{\Omega_t} [D_t, A]v \cdot Av dV \lesssim \int_{\Omega_t} |\nabla v| \cdot |Av|^2 dx \le |\nabla v|_{L^2} \cdot |Av|_{L^4}^2 \\
\lesssim \quad & |\nabla v|_{L^2} \cdot |Av|_{L^2}^{\frac{1}{2}} \cdot |\nabla (Av)|_{L^2}^{\frac{3}{2}} \lesssim |\nabla v|_{L^2}^4 \cdot |Av|_{L^2}^2 + \frac{1}{2} |\nabla (Av)|_{L^2}^2
\end{aligned}
$$

Since we have the boundary terms in the operator $A$, the commutator involves more terms.

$$
[D_t, A]v = [D_t, -\mathbb{P}\triangle]v + [D_t, \nabla \mathscr{H}(n \cdot T. \cdot n)]v = (I) + (II)
$$

Since $\partial_t$ commutes with $\mathbb{P}\triangle$,

$$
(I) = [v \cdot \nabla, -\mathbb{P}\triangle]v = \mathbb{P}\triangle v \cdot \nabla v + \{-v \cdot \nabla(\mathbb{P}\triangle v) + \mathbb{P}(v \cdot \nabla \triangle v) + \mathbb{P}(2\nabla v \cdot \nabla\nabla v)\} = (III) + (IV)
$$

The second term $(II)$ involves more commutators.

$$
\begin{aligned}
(II) &= D_t(\nabla \mathscr{H}(n \cdot T \cdot n)) - \nabla \mathscr{H}(n \cdot T_{D_t v} \cdot n) \\
&= \nabla D_t \mathscr{H}(n \cdot T \cdot n) + [D_t, \nabla]\mathscr{H}(n \cdot T \cdot n) - \nabla \mathscr{H}(n \cdot T_{D_t v} \cdot n) \\
&= \nabla \mathscr{H}(D_t(n \cdot T \cdot n)) + \nabla [D_t, \mathscr{H}](n \cdot T \cdot n) + [D_t, \nabla]\mathscr{H}(n \cdot T \cdot n) - \nabla \mathscr{H}(n \cdot T_{D_t v} \cdot n)
\end{aligned}
$$

The operator $D_t$ on the boundary is understood as

$$
D_t(RF) = [\frac{\partial}{\partial t}\{RF \circ u\}] \circ u^{-1} = \frac{\partial}{\partial t}(RF) + v \cdot \frac{\partial}{\partial u}(RF)
$$

where $RF$ is the restriction of $F$ onto the free surface $S_F$ and $u$ is the Lagrangian coordinate map solving $\frac{dx}{dt} = v(t,x), \quad x(0) = y$. Since $D_t$ is a linear first order differential operator on functions defined on the boundary and $D_t n$ is orthogonal to $n$, by the tangential boundary condition,

$$
\begin{aligned}
& D_t(n \cdot T \cdot n) = D_t n \cdot T \cdot n + n \cdot D_t T \cdot n + n \cdot T \cdot D_t n = n \cdot D_t T \cdot n = n_i n_j D_t R(v_{i,j} + v_{j,i}) \\
= \quad & n_i n_j R(D_t(v_{i,j} + v_{j,i})) + n_i n_j [\partial_t, R](v_{i,j} + v_{j,i}) + n_i n_j [v \cdot \nabla, R](v_{i,j} + v_{j,i}) \\
= \quad & n_i n_j \{((D_t v)_{i,j} + (D_t v)_{j,i}) + ([D_t, \partial_j]v_i + [D_t, \partial_i]v_j) + \partial_t \eta R(\partial_z(v_{i,j} + v_{j,i})) + [v \cdot \nabla, R](v_{i,j} + v_{j,i})\} \\
= \quad & n \cdot T_{D_t v} \cdot n + n_i n_j ([D_t, \partial_j]v_i + [D_t, \partial_i]v_j) + n_i n_j \partial_t \eta R(\partial_z(v_{i,j} + v_{j,i})) \\
+ \quad & n_i n_j \{R v_i \partial_i \eta R(\partial_z(v_{i,j} + v_{j,i})) + R v_3 R(\partial_z(v_{i,j} + v_{j,i})))\}
\end{aligned}
$$

After reordering terms,

$$
\begin{aligned}
(II) &= \nabla \mathscr{H}(n_i n_j ([D_t, \partial_j]v_i + [D_t, \partial_i]v_j)) + \nabla [D_t, \mathscr{H}](n \cdot T \cdot n) + [D_t, \nabla]\mathscr{H}(n \cdot T \cdot n) \\
&+ \nabla \mathscr{H}(n_i n_j \partial_t \eta R(\partial_z(v_{i,j} + v_{j,i})) + n_i n_j \{R v_i \partial_i \eta R(\partial_z(v_{i,j} + v_{j,i})) - R v_3 R(\partial_z(v_{i,j} + v_{j,i})))\}) \\
&= (V) + (VI) + (VII) + (VIII)
\end{aligned}
$$

By the identity used in Shatah-Zeng [10],

$$(VI) = \nabla(\triangle)^{-1}(2\partial v \cdot \nabla^2 \mathscr{H}(n \cdot T \cdot n) + \nabla \mathscr{H}(n \cdot T \cdot n) \cdot \triangle v) = \nabla(\triangle)^{-1}\triangle(v \cdot \nabla \mathscr{H}(n \cdot T \cdot n))$$

which is not reduced into $\nabla(v \cdot \nabla \mathscr{H}(n \cdot T \cdot n))$ because $\triangle$ is not invertible. Here, $\triangle^{-1}$ denotes the inverse of the Laplacian with zero Dirichlet boundary condition.

$$(VII) = -\nabla v \cdot \nabla \mathscr{H}(n \cdot T \cdot n)$$

where $(\cdot)$ is the matrix multiplication with a vector, not an inner product.

$$(III) + (VII) = -\nabla v \cdot (-\triangle v + \nabla \mathscr{H}(n \cdot T \cdot n)) = -\nabla v \cdot Av$$

From now on, we will apply Lemma 1 to lemma 4 and corollary 1 to obtain bounds.

• **Estimation of** $(III) + (VII)$**:** See (▶)

• **Estimation of** $(IV)$**:** It is almost the same as (▶) but it involves more terms because we $\nabla^2$ instead of $Av$. Since $\mathbb{P}$ is a bounded operator in $H^1$,

$$\begin{aligned}
&\int_{\Omega_t} Av \cdot (IV) dV \lesssim \int_{\Omega_t} |Avv| \cdot \{|v \cdot \nabla(\mathbb{P}\triangle v)| + |\mathbb{P}(v \cdot \nabla \triangle v)| + |\mathbb{P}(2\nabla v \cdot \nabla\nabla v)|\} dV \\
\lesssim\ & |Av|_{L^2} \cdot |\nabla v \nabla^2 v|_{L^2} + |Av|_{L^2} \cdot |v|_{L^\infty} \cdot (|\nabla^3 v|_{L^2} + |\nabla^2 v|_{L^2}) \\
\lesssim\ & |Av|_{L^2} \cdot |\nabla v|_{L^4} \cdot |\nabla^2 v|_{L^4} + |Av|_{L^2} \cdot (|v|_{L^2} + |Av|_{L^2}) \cdot (|\nabla Av|_{L^2} + |\nabla v|_{L^2} + |Av|_{L^2}) \\
\lesssim\ & |Av|_{L^2}^2 \cdot |\nabla v|_{L^2}^{\frac{1}{2}} \cdot |\nabla^3 v|_{L^2}^{\frac{3}{4}} + |v|_{L^2}^2 \cdot |Av|_{L^2}^2 + |\nabla v|_{L^2}^2 + |Av|_{L^2} \cdot |\nabla v|_{L^2}^2 \\
\lesssim\ & |Av|_{L^2}^2 \cdot |\nabla v|_{L^2} + |Av|_{L^2}^2 \cdot |\nabla v|_{L^2}^{\frac{1}{4}} \cdot |\nabla Av|_{L^2}^{\frac{3}{4}} + |v|_{L^2}^2 \cdot |Av|_{L^2}^2 + |\nabla v|_{L^2}^2 + |Av|_{L^2} \cdot |\nabla v|_{L^2}^2 \\
\lesssim\ & \frac{1}{2}|\nabla Av|_{L^2}^2 + |\nabla v|_{L^2}^4 + |Av|_{L^2}^{\frac{16}{5}} \cdot |\nabla v|_{L^2}^{\frac{2}{5}} + |v|_{L^2}^2 \cdot |Av|_{L^2}^2 + |\nabla v|_{L^2}^2 + |Av|_{L^2} \cdot |\nabla v|_{L^2}^2
\end{aligned}$$

• **Estimation of** $(V)$

$$(V) = \nabla \mathscr{H}(n_i n_j([D_t, \partial_j] v_i + [D_t, \partial_i] v_j)) = -\nabla \mathscr{H}(n_i n_j(\partial_j v \cdot \nabla v_i + \partial_i v \cdot \nabla v_j))$$

Then, by the integration by parts,

$$\begin{aligned}
&\int_{\Omega_t} Av \cdot (V) dV = \int_{\partial\Omega_t} (Av \cdot n)(n_i n_j(\partial_j v \cdot \nabla v_i + \partial_i \cdot \nabla v_j)) dS \\
\lesssim\ & |n \cdot Av|_{L^2(\partial\Omega_t)} \cdot |(n_i n_j(\partial_j v \cdot \nabla v_i + \partial_i \cdot \nabla v_j))|_{L^2(\partial\Omega_t)} \\
\lesssim\ & |\nabla(Av)|_{L^2} \cdot |\nabla v \cdot \partial v|_{H^1} + |Av|_{L^2} \cdot |\nabla v \cdot \nabla v|_{H^1} = ⓐ + ⓑ
\end{aligned}$$

$$
\begin{aligned}
ⓐ \lesssim\ & |\nabla(Av)|_{L^2}\cdot\{|\nabla v\cdot\nabla v|_{L^2}+|\nabla(\nabla v\cdot\nabla v)|_{L^2}\}\\
\lesssim\ & |\nabla(Av)|_{L^2}\cdot|\nabla v|_{L^2}\cdot|\nabla v|_{L^\infty}+|\nabla(Av)|_{L^2}\cdot|\nabla^2 v|_{L^4}\cdot|\nabla v|_{L^4}\\
\lesssim\ & |\nabla Av|_{L^2}\cdot|\nabla v|_{L^2}\cdot(|\nabla v|_{L^2}+|\nabla^3 v|_{L^2})+|\nabla Av|_{L^2}\cdot|\nabla^2 v|_{L^2}\cdot|\nabla v|^{\frac14}_{L^2}\cdot|\nabla^3 v|^{\frac34}_{L^2}\\
\lesssim\ & |\nabla Av|_{L^2}\cdot|\nabla v|^2_{L^2}+|\nabla Av|^2_{L^2}\cdot|\nabla v|_{L^2}+\nabla Av|^{\frac74}_{L^2}\cdot|\nabla^2 v|_{L^2}\cdot|\nabla v|^{\frac14}_{L^2}\\
\lesssim\ & \frac12|\nabla Av|^2_{L^2}+|\nabla v|^4_{L^2}+|\nabla Av|^2_{L^2}\cdot|\nabla v|_{L^2}+|\nabla^2 v|^8_{L^2}\cdot|\nabla v|^2_{L^2}+|\nabla^2 v|^2_{L^2}\cdot|\nabla v|^2_{L^2}\\
\lesssim\ & \frac12|\nabla Av|^2_{L^2}+|\nabla v|^4_{L^2}+|\nabla Av|^2_{L^2}\cdot|\nabla v|_{L^2}+|Av|^8_{L^2}\cdot|\nabla v|^2_{L^2}+|Av|^2_{L^2}\cdot|\nabla v|^2_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
ⓑ \lesssim\ & |Av|_{L^2}\cdot\{|\nabla v|_{H^1}\cdot|\nabla v|_{L^\infty}\}\\
\lesssim\ & |Av|_{L^2}\cdot|\nabla v|_{L^2}\cdot(|\nabla^3 v|_{L^2}+|\nabla v|_{L^2})+|Av|_{L^2}\cdot|\nabla^2 v|_{L^2}\cdot(|\nabla^3 v|_{L^2}+|\nabla v|_{L^2})\\
\lesssim\ & |Av|_{L^2}\cdot|\nabla v|_{L^2}\cdot|\nabla Av|_{L^2}+|Av|_{L^2}\cdot|\nabla v|^2_{L^2}+|Av|_{L^2}\cdot|\nabla^2 v|_{L^2}\cdot|\nabla Av|_{L^2}\\
\lesssim\ & \frac12|\nabla Av|^2_{L^2}+|Av|^2_{L^2}\cdot(|Av|_{L^2}+|\nabla v|_{L^2})^2+(|Av|_{L^2}+|\nabla v|_{L^2})^3
\end{aligned}
$$

• **Estimation of** $(VIII)$

$$
\begin{aligned}
& \int_{\Omega_t} Av\cdot(VIII)dV\\
=\ & \int_{\partial\Omega_t}(n\cdot Av)n_in_j\eta_t R(\partial_z(v_{i,j}+v_{j,i}))dS+\int_{\partial\Omega_t}(n\cdot Av)n_in_jRv_i\partial_i\eta R(\partial_z(v_{i,j}+v_{j,i}))dS\\
-\ & \int_{\partial\Omega_t}(n\cdot Av)n_in_jRv_3R(\partial_z(v_{i,j}+v_{j,i}))dS\\
\lesssim\ & |Av|_{H^1}\cdot|v\cdot\nabla^2 v|_{H^1}\lesssim(|Av|_{L^2}+|\nabla Av|_{L^2})(|v\cdot\nabla^2 v|_{L^2}+|\nabla(v\cdot\nabla^2 v)|_{L^2})\\
\lesssim\ & (|Av|_{L^2}+|\nabla Av|_{L^2})(|v\cdot\nabla^2 v|_{L^2}+|\nabla v\cdot\nabla^2 v|_{L^2}+|v\cdot\nabla^3 v|_{L^2}\cdot)\\
=\ & |Av|_{L^2}\cdot(|v\cdot\nabla^2 v|_{L^2}+|\nabla v\cdot\nabla^2 v|_{L^2}+|v\cdot\nabla^3 v|_{L^2})\\
+\ & |\nabla Av|_{L^2}\cdot(|v\cdot\nabla^2 v|_{L^2}+|\nabla v\cdot\nabla^2 v|_{L^2}+|v\cdot\nabla^3 v|_{L^2})=ⓒ+ⓓ
\end{aligned}
$$

$$
\begin{aligned}
ⓒ \lesssim\ & |Av|^2_{L^2}\cdot|v|_{L^\infty}+|Av|^2_{L^2}\cdot|\nabla v|_{L^\infty}+|Av|_{L^2}\cdot|v|_{L^\infty}\cdot|\nabla^3 v|_{L^2}\\
\lesssim\ & |Av|^2_{L^2}\cdot|v|_{L^2}+|Av|^3_{L^2}+|Av|^2_{L^2}\cdot|\nabla v|_{L^2}+|Av|^2_{L^2}\cdot|\nabla^3 v|_{L^2}\\
+\ & |Av|_{L^2}\cdot(|v|_{L^2}+|Av|_{L^2})(|\nabla v|_{L^2}+|\nabla Av|_{L^2})\\
\lesssim\ & |Av|^2_{L^2}\cdot(|v|_{L^2}+|Av|_{L^2}+|\nabla v|_{L^2}+|\nabla Av|_{L^2})+|Av|_{L^2}\cdot|v|_{L^2}\cdot(|\nabla v|_{L^2}+|\nabla Av|_{L^2})\\
\lesssim\ & |\nabla v|_{L^2}\cdot|\nabla Av|_{L^2}\cdot(|Av|_{L^2}+|v|_{L^2}+|\nabla v|_{L^2}+|\nabla Av|^2_{L^2}+|v|^2_{L^2})\\
+\ & |\nabla v|^{\frac32}_{L^2}\cdot|v|_{L^2}\cdot|\nabla Av|^{\frac12}_{L^2}+\frac12|\nabla Av|^2_{L^2}\\
\lesssim\ & |v|^2_{L^2}\cdot|\nabla v|^2_{L^2}+|Av|^2_{L^2}\cdot|\nabla v|^2_{L^2}+|\nabla v|^4_{L^2}+|\nabla v|_{L^2}\cdot|\nabla Av|^2_{L^2}+|v|^{\frac45}_{L^2}\cdot|\nabla v|^2_{L^2}\\
+\ & |v|^4_{L^2}\cdot|\nabla v|^2_{L^2}+\frac12|\nabla Av|^2_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
ⓓ &\lesssim |\nabla Av|_{L^2}\cdot|v|_{L^\infty}\cdot|\nabla^2 v|_{L^2}+|\nabla Av|_{L^2}\cdot|\nabla v|_{L^\infty}\cdot|\nabla^2 v|_{L^2}+|\nabla Av|_{L^2}\cdot|v|_{L^\infty}\cdot|\nabla^3 v|_{L^2}\\
&\lesssim |\nabla Av|_{L^2}\cdot(|v|_{L^2}+|Av|_{L^2}+|\nabla v|_{L^2}+|\nabla^3 v|_{L^2})\cdot|Av|_{L^2}\\
&+ |\nabla Av|_{L^2}\cdot(|v|_{L^2}+|Av|_{L^2})(|\nabla v|_{L^2}+|\nabla Av|_{L^2})\\
&\lesssim |v|_{L^2}^2\cdot|\nabla v|_{L^2}^2+|\nabla v|_{L^2}^4+|Av|_{L^2}^2\cdot|\nabla v|_{L^2}^2+|v|_{L^2}^2\cdot|\nabla v|_{L^2}^2+(|v|_{L^2}+|Av|_{L^2})\cdot|\nabla Av|_{L^2}^2+\frac{1}{2}|\nabla Av|_{L^2}^2
\end{aligned}
$$

• **Estimation of** $(VI)$**:** We can use Lemma 5.5. We solve the elliptic equation :

$$
\begin{cases}
-\triangle w = 2\partial v\cdot\nabla^2\mathscr{H}(n\cdot T\cdot n)+\nabla\mathscr{H}(n\cdot T\cdot n)\cdot\triangle v \quad \text{in} \quad \Omega\\
w=0 \quad \text{on} \quad \partial\Omega
\end{cases}
$$

Since $w=0$ at the bottom, we can use Poincare inequality to estimate the boundary term.

$$
\begin{aligned}
&\int_{\Omega_t} Av\cdot(VI)dV = \int_{\Omega_t} Av\cdot\nabla(\triangle)^{-1}\triangle(v\cdot\nabla\mathscr{H}(n\cdot T\cdot n))dV\\
\lesssim\ & |Av|_{L^2}\cdot|\nabla(\triangle)^{-1}\triangle(v\cdot\nabla\mathscr{H}(n\cdot T\cdot n))|_{L^2}\\
\lesssim\ & |Av|_{L^2}\cdot|2\partial v\cdot\nabla^2\mathscr{H}(n\cdot T\cdot n)+\nabla\mathscr{H}(n\cdot T\cdot n)\cdot\triangle v|_{L^2}\\
\lesssim\ & |Av|_{L^2}\cdot(|\partial v\cdot\nabla(\triangle v+Av)+\triangle v\cdot(\triangle v+Av)|_{L^2})\\
\lesssim\ & |Av|_{L^2}\cdot\{|\nabla v|_{L^\infty}\cdot(|\nabla^3 v|_{L^2}+|\nabla Av|_{L^2})+|\nabla^2 v|_{L^4}^2+|\nabla^2 v|_{L^4}\cdot|Av|_{L^4}\}\\
\lesssim\ & |Av|_{L^2}\cdot(|\nabla v|_{L^2}+|\nabla Av|_{L^2})^2+|Av|_{L^2}^{\frac{3}{2}}\cdot|\nabla Av|_{L^2}^{\frac{3}{2}}+|Av|_{L^2}^{\frac{3}{2}}\cdot|\nabla v|_{L^2}^{\frac{3}{4}}\cdot|\nabla Av|_{L^2}^{\frac{3}{4}}\\
\lesssim\ & |Av|_{L^2}\cdot(|\nabla v|_{L^2}+|\nabla Av|_{L^2})^2+|\nabla v|_{L^2}^3\cdot|Av|_{L^2}^3+|Av|_{L^2}^{\frac{12}{5}}\cdot|\nabla v|_{L^2}^{\frac{6}{5}}+\frac{1}{2}|\nabla Av|_{L^2}^2
\end{aligned}
$$

Collecting all terms, integrating in time, we finish proof of Proposition 2.2. ■

## 6. PROOF OF PROPOSITION 2.3, PROPOSITION 2.4 AND PROPOSITION 2.5

In this Chapter, we study the nonlinear term of $\eta$:

$$
\int_{\Omega_t} Av\cdot\nabla\mathscr{H}(n\cdot T_{\nabla\mathscr{H}(\eta-\nabla\cdot(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}))}\cdot n)dV = \frac{1}{2}\frac{d}{dt}\int_{R^2}(|\triangle_0\eta|^2+|\nabla\triangle_0\eta|^2)dxdy+(\alpha)
$$

Since we single out linear terms, $(\alpha)$ is cubic. As we did for the commutator term, we may expect that $\int(\alpha)dt \lesssim (||v||^2+|\eta|^2_{L_t^\infty H_x^3})^2+\frac{1}{2}(||v||^2+|\eta|^2_{L_t^\infty H_x^3})$.

▶ **Proposition 2.3. Estimate of** $(\alpha)$

$$
\int(\alpha)dt \lesssim |\eta|_{L_t^\infty H_x^3}\cdot||v||^2+|\eta|^2_{L_t^\infty H_x^3}\cdot|v_t|^2_{L_t^2H_x^1}+\frac{1}{2}|v_t|^2_{L_t^2H_x^1}+\frac{1}{2}||v||^4+\frac{1}{2}||v||^2
$$

**Proof :** First, we write all terms in $(\alpha)$. As mentioned, we set $Bv = -\triangle v + \nabla \mathscr{H}(n \cdot T \cdot n)$.

$$
\begin{aligned}
(\alpha) = & \int_{\partial\Omega_t} (n \cdot Bv) n_k n_l \partial_k \partial_l \mathscr{H}(\eta - \triangle_0 \eta) dS + \int_{\partial\Omega_t} \frac{1}{\sqrt{1+|\nabla\eta|^2}} \triangle_0 (v \cdot \nabla \eta) \cdot \triangle_0 (\eta - \triangle_0 \eta) dS \\
+ & \int_{\partial\Omega_t} \frac{|\nabla\eta|^2}{1+|\nabla\eta|^2} (n \cdot Bv) \triangle_0 \mathscr{H}(\eta - \triangle_0 \eta) dS + \int_{\partial\Omega_t} \frac{\partial_1 \eta (Bv)_1 + \partial_2 (Bv)_2}{\sqrt{1+|\nabla\eta|^2}} \triangle_0 \mathscr{H}(\eta - \triangle_0 \eta) dS \\
- & \int_{\partial\Omega_t} \frac{\triangle_0 v_3}{\sqrt{1+|\nabla\eta|^2}} \{ 2\nabla\eta \cdot \nabla \partial_3 \mathscr{H}(\eta - \triangle_0\eta) + \triangle_0 \eta \partial_3 \mathscr{H}(\eta - \triangle_0 \eta) + \frac{1}{2} |\nabla\eta|^2 \partial_3 \partial_3 \mathscr{H}(\eta - \triangle_0 \eta) \} dS \\
- & \int_{\partial\Omega_t} \frac{\triangle_0 \mathscr{H}(\eta - \triangle_0 \eta)}{\sqrt{1+|\nabla\eta|^2}} \{ (2\nabla\eta \cdot \nabla \partial_3 v_3 + \triangle_0 \eta \partial_3 v_3 + \frac{1}{2} |\nabla\eta|^2 \partial_3 \partial_3 v_3) - (\partial_3 \partial_3 v_3 - \partial_3 \mathscr{H}(n \cdot T \cdot n)) \} dS \\
+ & \int_{\partial\Omega_t} (n \cdot Bv) n_i n_j \partial_i \partial_j \mathscr{H}(\triangle_0 \eta - \nabla \cdot (\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}})) dS
\end{aligned}
$$

where the first integral is summed up over $k = 1,2,3$, $l = 1,2,3$ except $k = l = 3$. We have to show that $(\partial_3 \partial_3 v_3 - \partial_3 \mathscr{H}(n \cdot T \cdot n))$ in the fourth line is quadratic.

$$
\partial_3 \partial_3 v_3 - \partial_3 \mathscr{H}(n \cdot T \cdot n) = (\partial_3 \partial_3 v_3 - \partial_3 \mathscr{H}(\partial_3 v_3)) + \partial_3 \mathscr{H}(\frac{|\nabla\eta|^2}{1+|\nabla\eta|^2} \partial_3 v_3) - \partial_3 \mathscr{H}(n_k n_l (v_{k,l} + v_{l,k}))
$$

We apply the Dirichlet-Neumann operator argument to $(\partial_3 \partial_3 v_3 - \partial_3 \mathscr{H}(\partial_3 v_3))$. Since $R\mathscr{H} = Id$,

$$
\partial_3 \partial_3 v_3 = \frac{G(\eta) R(\partial_3 v_3) + \nabla\eta \nabla (R(\partial_3 v_3))}{1+|\nabla\eta|^2}, \quad \partial_3 \mathscr{H}(\partial_3 v_3) = \frac{G(\eta) R(\partial_3 v_3) + \nabla\eta \nabla (R(\partial_3 v_3))}{1+|\nabla\eta|^2}
$$

Therefore, these two terms are cancelled and $(\partial_3 \partial_3 v_3 - \partial_3 \mathscr{H}(n \cdot T \cdot n))$ is quadratic with coefficient $\nabla\eta$. There are $\nabla\eta$ terms for each integral, so we can make all the quantities as small as we want. But, we have half more derivatives to the harmonic extension parts and half less derivatives to the velocity field parts. Since all terms only depend on $(x, y)$, we transform $\partial\Omega_t$ to $R^2$, move half derivative from harmonic extension parts to the velocity field parts, go back to the original equation and apply trace theorem. When we transform the boundary, factor $\sqrt{1+|\nabla\eta|^2}$ and its reciprocal appear. But, it does not generate any large quantities.

$$
\begin{aligned}
(\alpha) \lesssim & \ |\eta|_{H_x^3} \cdot (|\nabla v|_{L^2}^2 + |\nabla A v|_{L^2}^2) + \frac{1}{10} |\nabla \mathscr{H}(\eta - \triangle_0 \eta)|_{H^1}^2 \\
+ & \int_{\partial\Omega_t} (n \cdot Bv) n_i n_j \partial_i \partial_j \mathscr{H}(\triangle_0 \eta - \nabla \cdot (\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}})) dS \\
\lesssim & \ |\eta|_{H_x^3} \cdot (|\nabla v|_{L^2}^2 + |\nabla A v|_{L^2}^2) + \frac{1}{10} |\nabla \mathscr{H}(\eta - \triangle_0 \eta)|_{H^1}^2 + |\eta|_{H_x^3} \cdot |Bv|_{H^1}^2 \\
+ & \ (\frac{1}{|\eta|_{H_x^3}} |\nabla\eta|_{L^\infty}^2) \cdot |\nabla\eta|_{L^\infty}^2 \cdot |\nabla^{\frac{5}{2}} \eta|_{H^1}^2 \\
\lesssim & \ |\eta|_{H_x^3} \cdot (|\nabla v|_{L^2}^2 + |\nabla A v|_{L^2}^2) + \frac{1}{10} |\nabla \mathscr{H}(\eta - \triangle_0 \eta)|_{H^1}^2 + |\eta|_{H^3}^2 \cdot |\nabla^{\frac{5}{2}} \eta|_{H^1}^2
\end{aligned}
$$

where we have $\frac{1}{10}$ factor in front of the boundary term by Young's inequality. Therefore,

$$\int(\alpha)dt \lesssim |\eta|_{L_t^\infty H_x^3}\cdot(|\nabla v|^2_{L_t^2L_x^2}+|\nabla Av|^2_{L_t^2L_x^2})+\frac{1}{10}|\nabla\mathscr{H}(\eta-F(\eta))|^2_{L_t^2H_x^1}+|\eta|^2_{L^\infty H_x^3}\cdot|\nabla^{\frac{5}{2}}\eta|^2_{L_t^2H_x^1} \quad (12)$$

We want to remove $|\nabla\mathscr{H}(\eta-F(\eta))|^2_{L_t^2H_x^1}$ in $\int(\alpha)dt$. From the equation,

$$|\nabla\mathscr{H}(\eta-F(\eta))|^2_{L_t^2H_x^1}\lesssim|\mathbb{P}D_tv|^2_{L_t^2H_x^1}+|Av|^2_{L_t^2H_x^1}\lesssim|v_t|^2_{L_t^2H_x^1}+||v||^4+||v||^2 \quad (13)$$

We need to estimate $|v_t|^2_{L_t^2H_x^1}$. We have $||v||^2$ in the bounds of $|\nabla\mathscr{H}(\eta-F(\eta))|^2_{L_t^2H_x^1}$ and $|v_t|^2_{L_t^2H_x^1}$. So it seems that we cannot apply the contraction mapping theorem. But, in the expression $(\alpha)$, we have factor $\frac{1}{10}$. So, we can move this lower order term to the left hand side in the final energy bound.

► **Lemma 6.1:** $|v_t|^2_{L_t^2H_x^1}\lesssim\epsilon+||v||^4+||v||^2+||v||^2\cdot|\eta|^2_{L_t^\infty H_x^3}+||v||^2\cdot|v_t|^2_{L_t^2H_x^1}$.

**Proof:** Since $v_t=0$ at the bottom, $|v_t|_{L_t^2H_x^1}\lesssim\int<v_t,v_t>dt$. We take $D_t$ to the equation.

$$D_t(v_t+\mathbb{P}(v\cdot\nabla v))+A(v_t+\mathbb{P}(v\cdot\nabla v))+D_t\nabla\mathscr{H}(\eta-F(\eta))=-[D_tv,A]v-A(v\cdot\nabla v-\mathbb{P}(v\cdot\nabla v))$$

We multiply by $(v_t+\mathbb{P}(v\cdot\nabla v))$ and integrate in the spatial variables.

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|v_t+\mathbb{P}(v\cdot\nabla v)|^2_{L^2}+<v_t+\mathbb{P}(v\cdot\nabla v),v_t+\mathbb{P}(v\cdot\nabla v)>\\ \lesssim\ &\frac{1}{2}|v_t+\mathbb{P}(v\cdot\nabla v)|^2_{L^2}+\int_{\Omega_t}[D_t,A]v\cdot(v_t+\mathbb{P}(v\cdot\nabla v))dV+|\partial^2(v\cdot\nabla v)|^2_{L^2}+|D_t\nabla\mathscr{H}(\eta-F(\eta))|^2_{L^2}\end{aligned}$$

Now, we estimate $D_t\nabla\mathscr{H}(\eta-F(\eta))$.

$$\begin{aligned}D_t\nabla\mathscr{H}(\eta-F(\eta))&=\nabla\mathscr{H}(D_t(\eta-F(\eta)))+\nabla[D_t,\mathscr{H}](\eta-F(\eta))+[D_t,\nabla]\mathscr{H}(\eta-F(\eta))\\&=\nabla\mathscr{H}(D_t(\eta-F(\eta)))+\nabla(\triangle)^{-1}\triangle(v\cdot\nabla\mathscr{H}(\eta-F(\eta)))-\nabla v\cdot\nabla\mathscr{H}(\eta-F(\eta))\end{aligned}$$

As before, $\triangle^{-1}$ denotes the inverse of the Laplacian with zero Dirichlet boundary condition.

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|v_t+\mathbb{P}(v\cdot\nabla v)|^2_{L^2}+<v_t+\mathbb{P}(v\cdot\nabla v),v_t+\mathbb{P}(v\cdot\nabla v)>\\ \lesssim\ &\frac{1}{2}|v_t+\mathbb{P}(v\cdot\nabla v)|^2_{L^2}+\int_{\Omega_t}[D_t,A]v\cdot(v_t+\mathbb{P}(v\cdot\nabla v))dV+|A(v\cdot\nabla v-\mathbb{P}(v\cdot\nabla v))|^2_{L^2}\\ +\ &|\nabla\mathscr{H}(\partial_t(\eta-F(\eta)))|^2_{L^2}+|\nabla\mathscr{H}(v\cdot\nabla(\eta-F(\eta)))|^2_{L^2}+|\nabla(\triangle)^{-1}\triangle(v\cdot\nabla\mathscr{H}(\eta-F(\eta)))|^2_{L^2}\\ +\ &|\nabla v\cdot\nabla\mathscr{H}(\eta-F(\eta))|^2_{L^2}\end{aligned}$$

We can bound $|v_t+\mathbb{P}(v\cdot\nabla v)|^2_{L^2}$ by $|v_t|^2_{L^2}+||v||^4$. We can control $|\partial^2(v\cdot\nabla)|^2_{L^2}$ by $||v||^4$. Remaining terms are:

$$
\begin{aligned}
& |\nabla \mathscr{H}(v\cdot\nabla(\eta-F(\eta)))|_{L^2}^2+|\nabla(\triangle)^{-1}\triangle(v\cdot\nabla\mathscr{H}(\eta-F(\eta)))|_{L^2}^2+|\nabla\mathscr{H}(\partial_t(\eta-F(\eta)))|_{L^2}^2 \\
\lesssim\ & |v\cdot\nabla\mathscr{H}(\eta-F(\eta))|_{H^1}^2+|\nabla v\cdot\nabla\mathscr{H}(\eta-F(\eta))|_{L^2}^2+|\nabla\mathscr{H}(\partial_t(\eta-F(\eta)))|_{L^2}^2 \\
\lesssim\ & ||v||^2\cdot|\nabla\mathscr{H}(\eta-F(\eta))|_{H^1}^2+|\eta_t|_{L^2}^2+|\nabla^{\frac{5}{2}}\eta_t|_{L^2}^2
\end{aligned}
$$

Integrating in time,

$$
\begin{aligned}
& \int < v_t+\mathbb{P}(v\cdot\nabla v), v_t+\mathbb{P}(v\cdot\nabla v) > dt \lesssim \epsilon+\frac{1}{2}|v_t|_{L_t^2L_x^2}^2+||v||^4+||v||^2\cdot|\nabla\mathscr{H}(\eta-F(\eta))|_{H^1}^2 \\
+\ & |\eta_t|_{L_t^2L_x^2}^2+|\nabla^{\frac{5}{2}}\eta_t|_{L_t^2L_x^2}^2+\int\int_{\Omega_t}[D_tv,A]v\cdot(v_t+\mathbb{P}(v\cdot\nabla v))dVdt \\
\lesssim\ & \epsilon+\frac{1}{2}|v_t|_{L_t^2L_x^2}^2+||v||^4+||v||^2\cdot|\nabla\mathscr{H}(\eta-F(\eta))|_{H^1}^2+\int\int_{\Omega_t}[D_tv,A]v\cdot(v_t+\mathbb{P}(v\cdot\nabla v))dVdt \\
+\ & (1+|\eta|_{L_t^\infty H_x^3})^2\cdot(|\nabla v|_{L_t^2L_x^2}^2+|\nabla Av|_{L_t^2L_x^2}^2)+(|v|_{L_t^\infty L_x^2}^2+|Av|_{L_t^\infty L_x^2}^2)\cdot|\nabla^{\frac{5}{2}}\eta|_{L_t^2H_x^1}^2 \\
\lesssim\ & \epsilon+\frac{1}{2}|v_t|_{L_t^2L_x^2}^2+||v||^4+\frac{1}{2}|\nabla\mathscr{H}(\eta-F(\eta))|_{L_t^2H_x^1}^2+|\eta|_{L_t^\infty H_x^3}^2\cdot||v||^2+||v||^2+||v||^2\cdot|\nabla^{\frac{5}{2}}\eta|_{L_t^2H_x^1}^2 \\
+\ & \int\int_{\Omega_t}[D_tv,A]v\cdot(v_t+\mathbb{P}(v\cdot\nabla v))dVdt
\end{aligned}
$$

Finally, we estimate $\int\int_{\Omega_t}[D_tv,A]v\cdot(v_t+\mathbb{P}(v\cdot\nabla v))dVdt$. We already know the explicit expression of $[D_t,A]v$ in the proof of Proposition 2.2. By the divergence free condition of $v_t+\mathbb{P}(v\cdot\nabla v)$, we do the same estimate by replacing $Av$ with $v_t+\mathbb{P}(v\cdot\nabla v)$ in the proof of Proposition of 2.2. Up to signs and $(n_in_j)$,

$$
\begin{aligned}
& \int[D_tv,A]v\cdot(v_t+\mathbb{P}(v\cdot\nabla v))dV=\int(\nabla v\cdot Av)\cdot(v_t+\mathbb{P}(v\cdot\nabla v))dV \\
+\ & \int(\mathbb{P}(\nabla v\cdot\nabla^2v)+\mathbb{P}(v\cdot\nabla\triangle v)+v\cdot\nabla(\mathbb{P}(\triangle v)))\cdot(v_t+\mathbb{P}(v\cdot\nabla v))dV \\
+\ & \int_{\partial\Omega_t}n\cdot(v_t+\mathbb{P}(v\cdot\nabla v))(\nabla v)^2dS+\int_{\partial\Omega_t}n\cdot(v_t+\mathbb{P}(v\cdot\nabla v))(\eta_t\nabla^2v+v\nabla\eta\nabla^2v+v\nabla^2v)dS \\
+\ & \int(v_t+\mathbb{P}(v\cdot\nabla v))\cdot(\nabla(\triangle)^{-1}\triangle(v\cdot\nabla\mathscr{H}(n\cdot T\cdot n)))dV
\end{aligned}
$$

Integrating in time,

$$
\int\int[D_tv,A]v\cdot(v_t+\mathbb{P}(v\cdot\nabla v))dVdt\lesssim\frac{1}{2}|v_t+\mathbb{P}(v\cdot\nabla v)|_{L_t^2L_x^2}^2+\frac{1}{2}|\nabla(v_t+\mathbb{P}(v\cdot\nabla v))|_{L_t^2L_x^2}^2+||v||^4
$$

Collecting all terms,

$$
\begin{aligned}
& \int < v_t+\mathbb{P}(v\cdot\nabla v), v_t+\mathbb{P}(v\cdot\nabla v) > dt \\
\lesssim\ & \epsilon+\frac{1}{2}|\nabla\mathscr{H}(\eta-F(\eta))|_{L_t^2H_x^1}^2+||v||^4+|\eta|_{L_t^\infty H_x^3}^2\cdot||v||^2+||v||^2+||v||^2\cdot|\nabla^{\frac{5}{2}}\eta|_{L_t^2H_x^1}^2 \qquad (14)
\end{aligned}
$$

Since $|\nabla^{\frac{5}{2}}\eta|^2_{L^2_tH^1_x} \lesssim |\nabla\mathscr{H}(\eta - F(\eta))|^2_{L^2_tH^1_x}$, (See Lemma 5.2), (12) implies that

$$|v_t|^2_{L^2_tH^1_x} \lesssim \epsilon + ||v||^4 + ||v||^2 + |\eta|^2_{L^\infty_tH^3_x} \cdot ||v||^2 + ||v||^2 \cdot |v_t|^2_{L^2_tH^1_x} \tag{15}$$

We finish proof of Lemma 6.1. ■

► **Lemma 6.2:** $|\nabla(\eta - F(\eta))|^2_{L^2_tH^{\frac{1}{2}}_x} \lesssim |\nabla\mathscr{H}(\eta - F(\eta))|^2_{L^2_tH^1_x}$

**Proof:** We will use the Dirichlet-Neumann operator. Suppose $\nabla\mathscr{H}(\eta - F(\eta)) \in L^2_tH^1_x$. We set $f = \eta - F(\eta)$. For $i = 1, 2$,

$$R(\partial_i\mathscr{H}(f)) = \partial_i(f) - \partial_i\eta\frac{G(\eta)f + \nabla\eta\cdot\nabla f}{1 + |\nabla\eta|^2} \in L^2_tH^{\frac{1}{2}}_x$$

Therefore, by the product rule of fractional derivatives, $|\nabla f|_{L^2_tH^{\frac{1}{2}}_x} \lesssim |\eta|_{L^\infty_tH^3_x}\cdot|\nabla f|_{L^2_tH^{\frac{1}{2}}_x} + |\nabla\mathscr{H}(\eta - F(\eta))|_{L^2_tH^1_x}$. Here, we use the fact that $G(\eta)$ is a first order pseudo-differential operator. By the smallness of $|\eta|_{L^\infty_tH^3_x}$, we finish proof of Lemma 6.2. ■

Now, we prove Proposition 2.4. Let us go back to the energy bound (6) in Chapter 2. From the commutator estimate,

$$||v||^2 + |\eta|^2_{L^\infty_tH^3_x} \lesssim \epsilon + ||v||^4 + |\int(\alpha)dt|$$

We replace $|\int(\alpha)dt|$ with (12).

$$\begin{aligned} ||v||^2 + |\eta|^2_{L^\infty_tH^3_x} &\lesssim \epsilon + ||v||^4 + \frac{1}{10}(|v_t|_{L^2_tH^1_x} + |\nabla\mathscr{H}(\eta - F(\eta))|^2_{L^2_tH^1_x}) \\ &+ |\eta|^2_{L^\infty_tH^3_x}\cdot||v||^2 + |\eta|^2_{L^\infty H^3_x}\cdot|\nabla\mathscr{H}(\eta - F(\eta))|^2_{L^2_tH^1_x} \end{aligned}$$

We substitute $||v||^2$ into (13).

$$\begin{aligned} |\nabla\mathscr{H}(\eta - F(\eta))|^2_{L^2_tH^1_x} &\lesssim \epsilon + ||v||^4 + \frac{1}{10}(|v_t|_{L^2_tH^1_x} + |\nabla\mathscr{H}(\eta - F(\eta))|^2_{L^2_tH^1_x}) \\ &+ |\eta|^2_{L^\infty_tH^3_x}\cdot||v||^2 + |\eta|^2_{L^\infty_tH^3_x}\cdot||v||^2 + |\eta|^2_{L^\infty H^3_x}\cdot|\nabla\mathscr{H}(\eta - F(\eta))|^2_{L^2_tH^1_x} \end{aligned}$$

We substitute $||v||^2$ into (15).

$$\begin{aligned} |v_t|^2_{L^2_tH^1_x} &\lesssim \epsilon + ||v||^4 + |\eta|^2_{L^\infty_tH^3_x}\cdot||v||^2 + ||v||^2\cdot|v_t|^2_{L^2_tH^1_x} + \frac{1}{10}(|v_t|_{L^2_tH^1_x} + |\nabla\mathscr{H}(\eta - F(\eta))|^2_{L^2_tH^1_x}) \\ &+ |\eta|_{L^\infty_tH^3_x}\cdot||v||^2 + |\eta|^2_{L^\infty H^3_x}\cdot|\nabla\mathscr{H}(\eta - F(\eta))|^2_{L^2_tH^1_x} \end{aligned}$$

By adding these two bounds,

$$\begin{aligned} |v_t|^2_{L^2_tH^1_x} + |\nabla\mathscr{H}(\eta - F(\eta))|^2_{L^2_tH^1_x} &\lesssim \epsilon + ||v||^4 + |\eta|_{L^\infty_tH^3_x}\cdot||v||^2 + |\eta|^2_{L^\infty_tH^3_x}\cdot||v||^2 \\ &+ ||v||^2\cdot|v_t|^2_{L^2_tH^1_x} + |\eta|^2_{L^\infty H^3_x}\cdot|\nabla\mathscr{H}(\eta - F(\eta))|^2_{L^2_tH^1_x} \end{aligned} \tag{16}$$

which is end of proof of Proposition 2.4. ■

By (16), we have the estimate of $|\int(\alpha)dt|$.

$$|\int(\alpha)dt| \lesssim |\eta|^2_{L^\infty_t H^3_x}\cdot ||v||^2 + ||v||^4 + ||v||^2\cdot |v_t|^2_{L^2_t H^1_x} + |\eta|^2_{L^\infty_t H^3_x}\cdot |\nabla\mathscr{H}(\eta - F(\eta))|^2_{L^2_t H^1_x}$$

Therefore, we finish proof of proposition 2.3. ■

**▶ Proposition 2.5. Korn-type inequality**

$$|\nabla Av|^2_{L^2_t L^2_x} \lesssim \int < Av, Av > dt + ||v||^4 + \frac{1}{2}|\nabla Av|^2_{L^2_t L^2_x} + |\nabla\mathscr{H}(\eta - F(\eta))|^2_{L^2_t H^1_x}$$

**Proof :** Since $Av$ does not vanish at the bottom, we cannot apply Korn's inequality directly to $Av$. But we can use the equation itself to apply Korn's inequality.

$$Av + \nabla\mathscr{H}(\eta - F(\eta)) - v\cdot\nabla v + \mathbb{P}(v\cdot\nabla v) = -v_t - v\cdot\nabla v$$

Since the right-hand side vanishes at the bottom,

$$\begin{aligned} &|\partial(Av + \nabla\mathscr{H}(\eta - F(\eta)) - v\cdot\nabla v + \mathbb{P}(v\cdot\nabla v))|^2_{L^2} \\ \lesssim\ & < Av + \nabla\mathscr{H}(\eta - F(\eta)) - v\cdot\nabla v + \mathbb{P}(v\cdot\nabla v), Av + \nabla\mathscr{H}(\eta - F(\eta)) - v\cdot\nabla v + \mathbb{P}(v\cdot\nabla v) > \end{aligned}$$

Therefore,

$$|\nabla Av|^2_{L^2 L^2} \lesssim \int < Av, Av > dt + ||v||^4 + \frac{1}{2}|\nabla Av|^2_{L^2 L^2} + |\nabla^2\mathscr{H}(\eta - F(\eta))|^2_{L^2 L^2}$$

This is the end of proof of Proposition 2.5. ■

## 7. CHANGE OF VARIABLES

In this Chapter, we present details of the change of variables which is used in Chapter 3. We define $\theta(t) : \Omega = \{(x_1, x_2, y); -1 < y < 0\} \to \{(x_1, x_2, z'); -1 < z' < \eta(x_1, x_2, t)\}$ by $\theta(x_1, x_2, y, t) = (x_1, x_2, \bar{\eta}(x_1, x_2, t) + y(1 + \bar{\eta}(x_1, x_2, t)))$, where $\bar{\eta}$ is the harmonic extension of $\eta$ into the fluid domain. By definition,

$$d\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ A & B & J \end{pmatrix}, \quad \zeta = (d\theta)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{A}{J} & -\frac{B}{J} & \frac{1}{J} \end{pmatrix}$$

where $A = (1+y)\bar{\eta}_{x_1}$, $B = (1+y)\bar{\eta}_{x_2}$, and $J = 1 + \bar{\eta} + \partial_y\bar{\eta}(1+y)$. We define $v$ on $\theta(\Omega)$ by $v_i = \frac{\theta_{i,j}}{J}w_j = \alpha_{ij}w_j$. Then, $v_1 = \frac{w_1}{J}$, $v_2 = \frac{w_2}{J}$, $v_3 = \frac{A}{J}w_1 + \frac{B}{J}w_2 + w_3$. We make replacements

$v_{i,j} = \zeta_{lj}\partial_l(\alpha_{ik}w_k)$. Then,

$$dv = (v_{i,j}) = \begin{pmatrix} \partial_1(\frac{1}{J}w_1) - \frac{1}{J}A\partial_3(\frac{1}{J}w_1) & \partial_2(\frac{1}{J}w_1) - \frac{1}{J}B\partial_3(\frac{1}{J}w_1) & \frac{1}{J}\partial_3(\frac{1}{J}w_1) \\ \\ \partial_1(\frac{1}{J}w_2) - \frac{1}{J}A\partial_3(\frac{1}{J}w_2) & \partial_2(\frac{1}{J}w_2) - \frac{1}{J}B\partial_3(\frac{1}{J}w_2) & \frac{1}{J}\partial_3(\frac{1}{J}w_2) \\ \\ v_{3,1} & v_{3,2} & v_{3,3} \end{pmatrix}$$

where $v_{3,1} = \partial_1(\frac{1}{J}Aw_1 + \frac{1}{J}Bw_2 + w_3) - \frac{1}{J}A\partial_3(\frac{1}{J}Aw_1 + \frac{1}{J}Bw_2 + w_3)$,
$v_{3,2} = \partial_2(\frac{1}{J}Aw_1 + \frac{1}{J}Bw_2 + w_3) - \frac{1}{J}B\partial_3(\frac{1}{J}Aw_1 + \frac{1}{J}Bw_2 + w_3)$, $\quad v_{3,3} = \frac{1}{J}\partial_3(\frac{1}{J}Aw_1 + \frac{1}{J}Bw_2 + w_3)$

First, we take time derivative.

$$v_{1,t} = \frac{1}{J}w_{1,t} + (\frac{1}{J})_{,t}w_1 + ((\theta)^{-1})_3'\partial_3(\frac{1}{J}w_1) = \frac{1}{J}w_{1,t} - \frac{1}{J^2}J_{,t}w_1 + ((\theta)^{-1})_3'\{\frac{1}{J}w_{1,3} - \frac{1}{J}J_{,3}w_1\}$$

$$v_{2,t} = \frac{1}{J}w_{2,t} + (\frac{1}{J})_{,t}w_2 + ((\theta)^{-1})_3'\partial_3(\frac{1}{J}w_2) = \frac{1}{J}w_{2,t} - \frac{1}{J^2}J_{,t}w_2 + ((\theta)^{-1})_3'\{\frac{1}{J}w_{2,3} - \frac{1}{J}J_{,3}w_2\}$$

$$\begin{aligned} v_{3,t} &= \frac{1}{J}Aw_{1,t} + (\frac{A}{J})_t w_1 + \frac{1}{J}Bw_{2,t} + (\frac{B}{J})_t w_2 + v_{3,t} + ((\theta)^{-1})_3'\partial_3(\frac{1}{J}Aw_1 + \frac{1}{J}Bw_2 + v_3) \\ &= \frac{1}{J}Aw_{1,t} - \frac{J_{,t}}{J^2}Aw_1 + \frac{1}{J}A_{,t}w_1 + \frac{1}{J}Bw_{2,t} - \frac{J_{,t}}{J^2}Bw_2 + \frac{1}{J}B_{,t}w_2 + w_{3,t} \\ &+ ((\theta)^{-1})_3'\{-\frac{1}{J^2}J_{,3}Aw_1 + \frac{1}{J}A_{,3}w_1 + \frac{1}{J}Aw_{1,3} + -\frac{1}{J^2}J_{,3}Bw_2 + \frac{1}{J}B_{,3}w_2 + \frac{1}{J}Bw_{2,3} + w_{3,3}\} \end{aligned}$$

Next,,we calculate the advection terms.

$v \cdot \nabla v_i = \frac{1}{J}w_1\partial_1(\frac{1}{J}w_i) + \frac{1}{J}w_2\partial_2(\frac{1}{J}w_i) + \frac{1}{J}w_3\partial_3(\frac{1}{J}w_i)$ for $i = 1, 2$

$v \cdot \nabla v_3 = \frac{1}{J}w_1\partial_1(\frac{A}{J}w_1 + \frac{B}{J}w_2 + w_3) + \frac{1}{J}w_2\partial_2(\frac{A}{J}w_1 + \frac{B}{J}w_2 + w_3) + \frac{1}{J}w_3\partial_3(\frac{A}{J}w_1 + \frac{B}{J}w_2 + w_3)$

Finally, we obtain the dissipation term. For the simplicity, we calculate $\triangle$ first.

$$\begin{aligned} \triangle &= \partial_{11} + \partial_{22} + \frac{1}{J}\partial_3(\frac{1}{J}\partial_3) \\ &- \partial_1(\frac{1}{J}A\partial_3) - \partial_2(\frac{1}{J}B\partial_3) - \frac{1}{J}A\partial_{31} + \frac{1}{J}A\partial_3(\frac{1}{J}A\partial_3) - \frac{1}{J}B\partial_{32} + \frac{1}{J}B\partial_3(\frac{1}{J}B\partial_3) \end{aligned}$$

Therefore,

$$\begin{aligned} \triangle v_i &= \partial_{11}(\frac{1}{J}w_i) + \partial_{22}(\frac{1}{J}w_i) + \frac{1}{J}\partial_3(\frac{1}{J}\partial_3(\frac{1}{J}w_i)) - \partial_1(\frac{1}{J}A\partial_3(\frac{1}{J}w_i)) - \partial_2(\frac{1}{J}B\partial_3(\frac{1}{J}w_i)) \\ &- \frac{1}{J}A\partial_{31}(\frac{1}{J}w_i) + \frac{1}{J}A\partial_3(\frac{1}{J}A\partial_3(\frac{1}{J}w_i)) - \frac{1}{J}B\partial_{32}(\frac{1}{J}w_i) + \frac{1}{J}B\partial_3(\frac{1}{J}B\partial_3(\frac{1}{J}w_i)) \end{aligned}$$

$$
\begin{aligned}
\triangle v_3 &= \partial_{11}(\frac{A}{J}w_1 + \frac{B}{J}w_2 + w_3) + \partial_{22}(\frac{A}{J}w_1 + \frac{B}{J}w_2 + w_3) + \frac{1}{J}\partial_3(\frac{1}{J}\partial_3(\frac{A}{J}w_1 + \frac{B}{J}w_2 + w_3)) \\
&- \partial_1(\frac{1}{J}A\partial_3(\frac{A}{J}w_1 + \frac{B}{J}w_2 + w_3)) - \partial_2(\frac{1}{J}B\partial_3(\frac{A}{J}w_1 + \frac{B}{J}w_2 + w_3)) - \frac{1}{J}A\partial_{31}(\frac{A}{J}w_1 + \frac{B}{J}w_2 + w_3) \\
&+ \frac{1}{J}A\partial_3(\frac{1}{J}A\partial_3(\frac{A}{J}w_1 + \frac{B}{J}w_2 + w_3)) - \frac{1}{J}B\partial_{32}(\frac{A}{J}w_1 + \frac{B}{J}w_2 + w_3) \\
&+ \frac{1}{J}B\partial_3(\frac{1}{J}B\partial_3(\frac{A}{J}w_1 + \frac{B}{J}w_2 + w_3))
\end{aligned}
$$

We define the pressure as $q = p \circ \theta$. Then, $\partial_1 p = \partial_1 q - \frac{1}{J}A\partial_3 q$, $\partial_2 p = \partial_2 q - \frac{1}{J}B\partial_3 q$, $\partial_3 p = \frac{1}{J}\partial_3 q$. We substitute all terms into the Navier-Stokes equations and its boundary conditions. Then we see the quadratic nonlinear terms mentioned in Chapter 3.

## APPENDIX: Local Well-posedness of the Free Boundary Value Problem of the Incompressible Navier-Stokes Equations Without Surface Tension

We study the incompressible Navier-Stokes equations with free boundary in $\Omega_t$ of finite depth without surface tension. We have the same equations and free boundary conditions with no surface tension.

$$
(NSF)\begin{cases} v_t + v\cdot\nabla v - \triangle v + \nabla p = 0 \quad \text{in} \quad \Omega_t \\ \nabla\cdot v = 0 \quad \text{in} \quad \Omega_t \\ v = 0 \quad \text{on} \quad S_B \\ \eta_t = v_3 - v_1\partial_x\eta - v_2\partial_y\eta \quad \text{on} \quad S_F \\ pn_i = (v_{i,j} + v_{j,i})n_j + \eta n_i \quad \text{on} \quad S_F \end{cases}
$$

Here, we want to prove the following theorem.

**THEOREM 3:** For arbitrary initial data $v_0 \in H^2$ and for sufficiently small initial data $\eta_0 \in H^{\frac{5}{2}}$, there is a finite $T < \infty$ such that there is a unique local in time solution $v$, $\eta$ and the pressure $p$. Moreover, the solution satisfies the energy bound: $|v|_{C_tH^2_x} + |v|_{L^2_tH^3_x} + |\eta|_{C_tH^{\frac{5}{2}}_x} + |\nabla p|_{L^2_tH^1_x} \lesssim C_0$, where $C_0 = |v_0|_{H^2} + |\eta_0|_{H^{\frac{5}{2}}}$.

**Remark:** Here, we obtain the continuity-in-time of the boundary because we solve the transport equation of $\eta$. ( Under the surface tension, we estimated the boundary by using the structure of the equation, not solving the transport equation. ) We need the smallness assumption to the initial profile of the boundary to show that the change of variables defines a diffeomorphism. If we use the Lagrangian coordinate to fix the boundary, it does not require the smallness assumption of the initial boundary data. But, we want to use the same method: we will solve the problem on the equilibrium domain.

### 1. A PRIORI ESTIMATE ON THE MOVING DOMAIN

In this chapter, we will obtain the a priori estimate on the moving domain. We will obtain exactly the same bounds of the velocity field as we obtained to the problem with surface tension.

**(1) BASIC ENERGY ESTIMATE**

We multiply by $v$ and integrate in the spatial variables. Then,

$$\begin{aligned} 0 &= \int_{\Omega_t} \frac{1}{2}\frac{d}{dt}|v|^2 dV + \int_{\Omega_t} \frac{1}{2}\nabla\cdot(v|v|^2)dV + \int_{\Omega_t}(-\triangle v)\cdot v dV + \int_{\Omega_t} \nabla p\cdot v dV \\ &= \frac{1}{2}\frac{d}{dt}\int_{\Omega_t}|v|^2 dV - \frac{1}{2}\int_{\partial\Omega_t}(v\cdot n)|v|^2 dS + \frac{1}{2}\int_{\partial\Omega_t}(v\cdot n)|v|^2 dS \\ &+ \frac{1}{2}\int_{\Omega_t}|v_{i,j}+v_{j,i}|^2 dV - \int_{\partial\Omega_t}(v_{i,j}+v_{j,i})n_j v_i dS + \int_{\partial\Omega_t} p n_i v_i dS \end{aligned}$$

Collecting terms, we obtain that

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}|v|^2 dV + \frac{1}{2}\int_{\Omega_t}|v_{i,j}+v_{j,i}|^2 dV + \int_{\partial\Omega_t}(v\cdot n)\eta dS = 0$$

Since $\eta_t = \sqrt{1+|\nabla\eta|^2}(v\cdot n)$, the above equation can be written as

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}|v|^2 dV + \frac{1}{2}\int_{\Omega_t}|v_{i,j}+v_{j,i}|^2 dV + \int_{\partial\Omega_t}\frac{\eta_t\eta}{\sqrt{1+|\nabla\eta|^2}}dS = 0$$

By the change of variables, $\int_{\partial\Omega_t}\frac{\eta_t\eta}{\sqrt{1+|\nabla\eta|^2}}dS = \int_{R^2}(\eta_t\eta)dxdy = \frac{1}{2}\frac{d}{dt}\int_{R^2}|\eta|^2 dxdy$. Therefore,

$$\frac{1}{2}\frac{d}{dt}\{\int_{\Omega_t}|v|^2 dV + \int_{R^2}|\eta|^2\} + \frac{1}{2}\int_{\Omega_t}|v_{i,j}+v_{j,i}|^2 dV = 0$$

Integrating in time,

$$\frac{1}{2}|v(t)|^2_{L^2} + \frac{1}{2}|\eta(t)|^2_{L^2} + \frac{1}{2}\int_0^t\int_{\Omega_s}|v_{i,j}+v_{j,i}|^2 dV ds = \frac{1}{2}|v_0|^2_{L^2} + \frac{1}{2}|\eta_0|^2_{L^2}$$

By Korn's inequality, $|v|^2_{L^\infty_t L^2_x} + |\nabla v|^2_{L^2_t L^2_x} + |\eta|^2_{L^\infty_t L^2_x} \lesssim |v_0|^2_{L^2} + |\eta_0|^2_{L^2}$

**(2) HIGHER ENERGY ESTIMATE**

We will use the same vector field decomposition method to rewrite the equations. We take $\mathbb{P}$ to the equation.

$$\mathbb{P}D_t v + Av + \nabla\mathscr{H}(\eta) = 0, \quad Aw = -\mathbb{P}\triangle w + \nabla\mathscr{H}(n\cdot T_v\cdot n)$$

Since $A$ does not commute with the projection $\mathbb{P}$ onto the divergence free space,

$$A(D_t v) + A(Av) + A(\nabla\mathscr{H}(\eta)) = -A\{v\cdot\nabla v - \mathbb{P}(v\cdot\nabla v)\}$$

By commuting $D_t$ with $A$,

$$D_t(Av) + A(Av) + A(\nabla\mathscr{H}(\eta)) = [D_t, A]v - A\{v\cdot\nabla v - \mathbb{P}(v\cdot\nabla v)\}$$

where $A(\nabla\mathscr{H}(\eta)) = -\mathbb{P}\triangle(\nabla\mathscr{H}(\eta)) + \nabla\mathscr{H}(n\cdot T_{\nabla\mathscr{H}(\eta)}\cdot n) = \nabla\mathscr{H}(n\cdot T_{\nabla\mathscr{H}(\eta)}\cdot n)$. We multiply the equation by $Av$, integrate in the spatial variables over $\Omega_t$, and integrate in time.

$$\begin{aligned} & |Av(t)|_{L^2}^2 + \int_0^t < Av, Av > ds + \int_0^t \int_{\Omega_s} \nabla\mathscr{H}(n\cdot T_{\nabla\mathscr{H}(\eta)}\cdot n)\cdot Av dV ds \\ \lesssim\ & C_0 + \int_0^t\int_{\Omega_s} [D_t, A]v\cdot Av dV ds + \int_0^t\int_{\Omega_s} Av\cdot A\{v\cdot\nabla v - \mathbb{P}(v\cdot\nabla v)\} dV ds \end{aligned}$$

First of all, we estimate $\int_0^t\int_{\Omega_s} Av\cdot\nabla\mathscr{H}(n\cdot T_{\nabla\mathscr{H}(\eta)}\cdot n)dVds$. By the integration by parts,

$$\begin{aligned} & \int_{\Omega_t} Av\cdot\nabla\mathscr{H}(n\cdot T_{\nabla\mathscr{H}(\eta)}\cdot n)dV = \int_{\partial\Omega_t}(n\cdot Av)(n\cdot T_{\nabla\mathscr{H}(\eta)}\cdot n)dS \\ \lesssim\ & |Av|_{H^1(\Omega_t)}|n\cdot T_{\nabla\mathscr{H}(\eta)}\cdot n|_{L^2(\partial\Omega_t)} \lesssim \{|Av|_{L^2(\Omega_t)} + |\nabla(Av)|_{L^2(\Omega_t)}\}\cdot|n\cdot T_{\nabla\mathscr{H}(\eta)}\cdot n|_{L^2(\partial\Omega_t)} \\ \lesssim\ & |\nabla v|_{L^2(\Omega_t)}^2 + \frac{1}{2}|\nabla(Av)|_{L^2(\Omega_t)}^2 + |n\cdot T_{\nabla\mathscr{H}(\eta)}\cdot n|_{L^2(\partial\Omega_t)}^2 \end{aligned}$$

Here, we used the trace theorem such that $H^1(\Omega)\to L^2(\partial\Omega)$ to the velocity field. If we use $H^{\frac{1}{2}}(\Omega)\to L^2(\partial\Omega)$, we only need to estimate the boundary in $L_t^2H_x^{\frac{3}{2}}$, not in $L_t^2H_x^2$ which is stated below. When we study the free boundary value problem with surface tension, this sharp estimate is necessary. We will establish $L_t^\infty$ bounds by solving a transport equation of $\eta$. We can obtain $L_t^2$ bounds of the boundary from $L_t^\infty$ bounds. Secondly, we estimate the commutator.

**▶ Proposition: Commutator estimate**

$$\int_0^T\int_{\Omega_s}[D_t, A]v\cdot Av dV dt \lesssim T\cdot(|v|_{L_t^\infty L_x^2}^2 + |\nabla v|_{L_t^2L_x^2}^2 + |Av|_{L_t^\infty L_x^2}^2 + |\nabla Av|_{L_t^2L_x^2}^2)^2 + \frac{1}{2}|\nabla(Av)|_{L_t^2L_x^2}^2$$

**Proof:** We have the same commutator estimate as we had to the problem with surface tension. Then, we replace any $L^2$ in time term with $\sqrt{T}|\cdot|_{L_t^\infty H_x^k}$ for some $k\ge 0$. In this way, we can generate factor $T$ to the right-hand side.

In sum, we have the following energy bound:

$$\begin{aligned} & |v|_{L_t^\infty L_x^2}^2 + |\nabla v|_{L_t^2L_x^2}^2 + |Av|_{L_t^\infty L_x^2}^2 + |\nabla Av|_{L_t^2L_x^2}^2 \\ \lesssim\ & C_0 + T\cdot(|v|_{L_t^\infty L_x^2}^2 + |\nabla v|_{L_t^2L_x^2}^2 + |Av|_{L_t^\infty L_x^2}^2 + |\nabla Av|_{L_t^2L_x^2}^2)^2 + |n\cdot T_{\nabla\mathscr{H}(\eta)}\cdot n|_{L_t^2L^2(\partial\Omega_t)}^2 \end{aligned}$$

We estimate $|n\cdot T_{\nabla \mathscr{H}(\eta)}\cdot n|^2_{L^2(\partial\Omega_t)}$. By calculation,

$$\begin{aligned} n\cdot T_{\nabla \mathscr{H}(\eta)}\cdot n &= \frac{\partial_i\eta\partial_j\eta\partial_{ij}\eta}{1+|\nabla\eta|^2}+\triangle_0\eta - R(\partial_i\partial_z\mathscr{H}(\eta))\times\frac{(3\partial_i\eta+4\partial_i\eta|\nabla\eta|^2)}{1+|\nabla\eta|^2} \\ &- R(\partial_z\mathscr{H}(\eta))\times\frac{(1+|\nabla\eta|^2)\triangle_0\eta+\partial_i\eta\partial_j\eta\partial_{ij}\eta}{1+|\nabla\eta|^2} \end{aligned}$$

Therefore,

$$|n\cdot T_{\nabla \mathscr{H}(\eta)}\cdot n|^2_{L^2_tL^2(\partial\Omega_t)} \lesssim \{1+|\nabla\eta|^2_{L^\infty_tL^\infty(R^2)}+|\nabla\eta|^4_{L^\infty_tL^\infty(R^2)}+|\nabla\eta|^6_{L^\infty_tL^\infty(R^2)}\}\cdot|\nabla^2\eta|^2_{L^2_tL^2(R^2)}$$

We need to obtain $|\nabla\eta|_{L^\infty_tL^\infty(R^2)}$ and $|\nabla^2\eta|_{L^2_tL^2(R^2)}$. $\eta$ satisfies a transport equation:

$$\eta_t+v\cdot\nabla\eta=v_3$$

Since $Rv$ in the advection term does not satisfy the divergence free condition, we need a higher regularity to $v$ than the Lipschitz regularity. Here, $H^{\frac{5}{2}}(\partial\Omega_t)$ is enough. See [6]. On time interval $[0,T]$,

$$|\eta|_{L^\infty_tH^{\frac{5}{2}}} \lesssim \{|\eta_0|_{H^{\frac{5}{2}}}+\sqrt{T}|v_3|_{L^2_tH^{\frac{5}{2}}_x}\}\exp(\sqrt{T}|v|_{L^2_tH^{\frac{5}{2}}_x})$$

By the change of variables, $|v|_{L^2_tH^{\frac{5}{2}}_x(R^2)} \lesssim |v|_{L^2_tH^{\frac{5}{2}}_x(\partial\Omega_t)}$. Taking $T$ small enough, assuming a priori $\sqrt{T}|v|_{L^2_tH^{\frac{5}{2}}_x}\lesssim\frac{1}{2}$,

$$|\eta|_{L^\infty_tH^{\frac{5}{2}}} \lesssim \{|\eta_0|_{H^{\frac{5}{2}}}+\sqrt{T}|v|_{L^2_tH^{\frac{5}{2}}_x}\}\exp(\sqrt{T}|v|_{L^2_tH^{\frac{5}{2}}_x}) \lesssim |\eta_0|_{H^{\frac{5}{2}}}+\sqrt{T}|v|_{L^2_tH^{\frac{5}{2}}_x}$$

Therefore, $|\nabla^2\eta|^2_{L^2_tL^2}\le|\nabla^2\eta|^2_{L^2_tH^{\frac{3}{2}}}\le T|\eta|^2_{L^\infty_tH^{\frac{5}{2}}}\lesssim|\eta_0|^2_{H^{\frac{5}{2}}}+T^2|v|^2_{L^2_tH^{\frac{5}{2}}_x}$. By taking $T$ small enough, we can move $T^2\cdot|v|^2_{L^2_tH^{\frac{5}{2}}_x}$ to the left-hand side.

As before, we replace $\int <Av,Av> dt$ with $|\nabla Av|^2_{L^2_tL^2_x}$ with $|\nabla\mathscr{H}(\eta)|^2_{L^2_tH^1_x}$ in the right-hand side. But, it is less than $|\eta_0|^2_{H^{\frac{5}{2}}}+T^2|v|^2_{L^2_tH^{\frac{5}{2}}_x}$. Therefore, we can move $T^2|v|^2_{L^2_tH^{\frac{5}{2}}_x}$ to the left-hand side by taking $T$ small enough. A priori estimate becomes

$$|v|^2_{L^\infty_tL^2_x}+|\nabla v|^2_{L^2_tL^2_x}+|Av|^2_{L^\infty_tL^2_x}+|\nabla Av|^2_{L^2_tL^2_x}\lesssim C_0+T\cdot(|v|^2_{L^\infty_tL^2_x}+|\nabla v|^2_{L^2_tL^2_x}+|Av|^2_{L^\infty_tL^2_x}+|\nabla Av|^2_{L^2_tL^2_x})^2$$

We define a norm $||v||$ as $||v||=|v|_{L^\infty_tL^2_x}+|\nabla v|_{L^2_tL^2_x}+|Av|_{L^\infty_tL^2_x}+|\nabla Av|_{L^2_tL^2_x}$. Then, we have that $||v||^2\lesssim C_0+T\cdot||v||^4$ which will imply the contraction of $v$ for short time $T$ in the iteration step. The size of time interval is given by $[0,T^\star]$, $T^\star\sim\min\{1,\frac{1}{C_0}\}$. This also implies that $|\eta|_{L^\infty_tH^{\frac{5}{2}}}\lesssim C_0$. After solving $v$ and $\eta$, we solve the pressure from the following elliptic equation.

$$\begin{cases} -\triangle p=\partial v\partial v & \text{in } \ \Omega_t \\ pn_i=(v_{i,j}+v_{j,i})n_j+\eta n_i & \text{on } \ \partial\Omega_t \end{cases}$$

Since we have estimates on the harmonic extension parts, we only need to solve the Lagrangian multiplier $\nabla p_{v,v}$. The existence and uniqueness of $\nabla p_{v,v}$ is proved by those of $v_t$. By taking $\mathbb{P}$ to the momentum equation, we can prove that $v_t$ exists uniquely in $L^2_t H^1_x$. From the equation, in the time interval $[0,T]$,

$$\begin{aligned}
|\nabla p_{v,v}|_{L^2_t H^1_x} &\lesssim |v_t|_{L^2_t H^1_x} + |v\cdot\nabla v|_{L^2_t H^1_x} + |Av|_{L^2_t H^1_x} + |\nabla \mathscr{H}(\eta)|_{L^2_t H^1_x} \\
&\lesssim |\mathbb{P}(v\cdot\nabla v)|_{L^2_t H^1_x} + |v\cdot\nabla v|_{L^2_t H^1_x} + |Av|_{L^2_t H^1_x} + |\nabla \mathscr{H}(\eta)|_{L^2_t H^1_x} \\
&\lesssim |v\cdot\nabla v|_{L^2_t H^1_x} + |Av|_{L^2_t H^1_x} + |\nabla \mathscr{H}(\eta)|_{L^2_t H^1_x} \lesssim C_0
\end{aligned}$$

## 2. ITERATION, EXISTENCE AND UNIQUENESS

In this section, we transform the domain to the equilibrium domain and do the iteration. As before, we define $\theta(t) : \Omega = \{(x,y,z); -1 < y < 0\} \to \{(x,y,z'); -1 < z < \eta(x,y,t)\}$ by

$$\theta(x,y,z,t) = (x,y,\bar{\eta}(x,y,t) + z(1-\bar{\eta}(x,y,t)))$$

where $\bar{\eta}$ is the harmonic extension of $\eta$ into the fluid domain. We cannot follow the change of variables used in the problem with surface tension because it requires that $\nabla^4\bar{\eta} \in L^2$, while we only have $\nabla^3\bar{\eta}$ in the a priori estimate. To avoid this higher regularity of $\eta$, we simply compose $v$ an $p$ with $\theta$ to define the equations on the flat domain $\Omega$. See [8]. We define the velocity field and the pressure on $\Omega$ as $w = v\circ\theta$, $q = p\circ\theta$. Let $X = \theta^{-1}$. Then, the Jacobian matrix $dX$ is given by

$$(X_{i,j}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{A}{J} & \frac{B}{J} & \frac{1}{J} \end{pmatrix}$$

where $A = \partial_1\bar{\eta} - z\partial_1\bar{\eta}$, $B = \partial_2\bar{\eta} - z\partial_2\bar{\eta}$, $J = 1 - \bar{\eta} + \partial_z\bar{\eta}(1-z)$. Note that $X = \theta^{-1}$ exists as long as $\bar{\eta}$ is small enough. Now,

$$v_{i,t} = w_{i,t} + w_{i,3}X_{3,t}, \quad v_{i,k} = X_{j,k}w_{i,j}, \quad v\cdot\nabla v = w_k X_{j,k} w_{i,j}, \quad \frac{\partial^2 v_i}{\partial^2 z} = X_{l,k}X_{j,k}w_{i,lj} + X_{l,kk}w_{i,l}$$

Then, on the equilibrium domain, we have the following system of equations.

$$(LNSF)\begin{cases} w_t - \triangle w + \nabla q = f \quad \text{in} \quad \Omega \\ \nabla\cdot w = \sigma \quad \text{in} \quad \Omega \\ w_{i,3} + w_{3,i} = g_i, \quad q = w_{3,3} + \eta + g_3 \quad \text{on} \quad \{z=0\} \\ w = w_0 = v_0\circ\theta_0, \quad \eta = \eta_0 \quad \text{at} \quad t=0 \\ \eta_t - w_3 = w\cdot\nabla\eta = h \quad \text{on} \quad \{z=0\}, \quad w = 0 \quad \text{on} \quad \{z=-1\} \end{cases}$$

where nonlinear terms are given as

$$f_i = 2\tfrac{A}{J}w_{i,13} + 2\tfrac{B}{J}w_{i,23} + (\tfrac{A^2}{J^2} + \tfrac{B^2}{J^2} + \tfrac{1}{J^2} - 1)w_{i,33} + \triangle X_3 w_{i,3} - w_k X_{j,k} w_{i,j} - X_{j,i}\partial_j q + \partial_i q - v_{i,3}X_{3,t}$$

$$\sigma = -\frac{A}{J}w_{i,3} - \frac{B}{J}w_{2,3} + (1-\frac{1}{J})w_{3,3}$$

$$g_1 = -(\partial_1\eta)(\eta - q + 2(w_{1,1} - w_{1,3}\frac{A}{J})) - (\partial_2\eta)(w_{1,2} + w_{1,3}\frac{B}{J} + w_{2,1} + w_{2,3}\frac{A}{J}) - (1-\frac{1}{J})w_{1,3} - w_{3,3}\frac{A}{J}$$

$$g_2 = -(\partial_1\eta)(w_{1,2} + w_{1,3}\frac{B}{J} + w_{2,1} + w_{2,3}\frac{A}{J}) - (\partial_2\eta)(\eta - q - 2(w_{2,2} - w_{2,3}\frac{B}{J})) - (1-\frac{1}{J})w_{2,3} - w_{3,3}\frac{B}{J}$$

$$g_3 = -(\partial_1\eta)(w_{1,3}\frac{1}{J} + w_{3,1} + w_{3,3}\frac{A}{J}) - (\partial_2\eta)(w_{2,3}\frac{1}{J} + w_{3,2} + w_{3,3}\frac{B}{J}) - 2(1-\frac{1}{J})w_{3,3}$$

The compatibility conditions are given by

$$\begin{cases} w_0 = 0 \quad \text{on} \quad S_B, \quad \nabla\cdot w_0 = \sigma(0) \quad \text{in} \quad \Omega \\ \{-(w_0)_{1,3} + (w_0)_{3,1}, -(w_0)_{2,3} - (w_0)_{3,2}, q(0) - 2(w_0)_{3,3} - \eta\}\cdot T^k(0) = g(0)\cdot T^k(0) \end{cases}$$

where $T^1 = \frac{1}{\sqrt{1+|\partial_1\eta|^2}}(1,0,\partial_1\eta), \quad T^2 = \frac{1}{\sqrt{1+|\partial_2\eta|^2}}(0,1,\partial_2\eta)$.

Since we lose the divergence free condition, we need to reformulate the system into the problem of the divergence free by introducing $v$ such that $\nabla\cdot v = \sigma$, $v = 0$ at the bottom. We will discuss the existence of such a $v$ later. Let $u = w - v$. Then, $(u, \eta, q)$ satisfies the following system of equations.

$$(LNSF)\begin{cases} u_t - \triangle u + \nabla q = f - v_t + \triangle v \quad \text{in} \quad \Omega \\ \nabla\cdot u = 0 \quad \text{in} \quad \Omega \\ u_{i,3} + u_{3,i} = g_i - v_{i,3} - v_{3,i}, \quad q = u_{3,3} + \eta + g_3 + v_{3,3} \quad \text{on} \quad \{z=0\} \\ u = w_0 - v_0 \quad \text{at} \quad t = 0 \\ \eta_t - u_3 = u\cdot\nabla\eta + v\cdot\nabla\eta + v_3 = h \quad \text{on} \quad \{z=0\}, \quad u = 0 \quad \text{on} \quad \{z=-1\} \end{cases}$$

Let $F = f - v_t + \triangle$, $G_i = g_i - v_{i,3} - v_{3,i}$, $G_3 = \eta_0 + g_3 + \int_0^t h ds$. Then, we can rewrite the above system only in terms of $u$.

$$(LNSF)\begin{cases} u_t - \triangle u + \nabla q = F \quad \text{in} \quad \Omega \\ \nabla\cdot u = 0 \quad \text{in} \quad \Omega \\ u_{i,3} + u_{3,i} = G_i, \quad q = u_{3,3} + \int_0^t u_3 ds + G_3 \quad \text{on} \quad \{z=0\} \\ u = w_0 - v_0 \quad \text{at} \quad t = 0, \quad u = 0 \quad \text{on} \quad \{z=-1\} \end{cases}$$

There exists a unique solution $(u,q)$ such that

$$
\begin{aligned}
& |u|_{L^2_tH^3_x} + |u_t|_{L^2_tH^1_x} + |\nabla q|_{L^2_tH^1_x} + |\int_0^t u_3 ds|_{L^\infty_t H^2_x} \lesssim C_0 + |\eta_0|_{H^2_x} + |F|_{L^2_tH^1_x} + |G|_{L^2_tH^{\frac{3}{2}}_x} \\
\lesssim\ & C_0 + |f|_{L^2_tH^1_x} + |g|_{L^2_tH^{\frac{3}{2}}_x} + |v_t - \triangle v|_{L^2_tH^1_x} + |v_{3,i} + v_{i,3}|_{L^2_tH^{\frac{3}{2}}_x} + |\int_0^t h ds|_{L^2_tH^{\frac{3}{2}}_x} \\
\lesssim\ & C_0 + |f|_{L^2_tH^1_x} + |g|_{L^2_tH^{\frac{3}{2}}_x} + |v|_{L^2_tH^3_x} + |v_t|_{L^2_tH^1_x} + |\int_0^t (u\cdot\nabla\eta) ds|_{L^2_tH^{\frac{3}{2}}_x} + |\int_0^t (v\cdot\nabla\eta) ds|_{L^2_tH^{\frac{3}{2}}_x} \\
\lesssim\ & C_0 + |f|_{L^2_tH^1_x} + |g|_{L^2_tH^{\frac{3}{2}}_x} + |v|_{L^2_tH^3_x} + |v_t|_{L^2_tH^1_x} + T\cdot|u|_{L^2_tH^2_x}\cdot|\eta|_{L^2_tH^{\frac{5}{2}}_x} + T\cdot|v|_{L^2_tH^2_x}\cdot|\eta|_{L^2_tH^{\frac{5}{2}}_x}
\end{aligned}
$$

Now, we go back to the original velocity field. Since $w = u + v$,

$$
\begin{aligned}
& |w|_{L^2_tH^3_x} + |w_t|_{L^2_tH^1_x} + |\nabla q|_{L^2_tH^1_x} \\
\lesssim\ & C_0 + |f|_{L^2_tH^1_x} + |g|_{L^2_tH^{\frac{3}{2}}_x} + |v|_{L^2_tH^3_x} + |v_t|_{L^2_tH^1_x} + T\cdot|w|_{L^2_tH^2_x}\cdot|\eta|_{L^2_tH^{\frac{5}{2}}_x} + T\cdot|v|_{L^2_tH^2_x}\cdot|\eta|_{L^2_tH^{\frac{5}{2}}_x}
\end{aligned}
$$

Since $|\eta|_{L^\infty_t H^{\frac{5}{2}}} \lesssim |\eta_0|_{H^{\frac{5}{2}}} + \sqrt{T}|v|_{L^2_tH^{\frac{5}{2}}_x}$,

$$
\begin{aligned}
& |w|_{L^2_tH^3_x} + |w_t|_{L^2_tH^1_x} + |\nabla q|_{L^2_tH^1_x} \\
\lesssim\ & C_0 + |f|_{L^2_tH^1_x} + |g|_{L^2_tH^{\frac{3}{2}}_x} + |v|_{L^2_tH^3_x} + |v_t|_{L^2_tH^1_x} + T\cdot|w|^2_{L^2_tH^2_x} + T\cdot|v|_{L^2_tH^2_x}\cdot|w|_{L^2_tH^3_x}
\end{aligned}
$$

By solving $v$, we obtain the energy bound of $w$. The iteration is performed in the following way

$$
(LNSF^m)\begin{cases}
w^m_t - \triangle w^m + \nabla q^m = f(w^{m-1},\eta^{m-1},q^{m-1}) \quad \text{in} \quad \Omega \\
\nabla\cdot w^m = \sigma(w^{m-1},\eta^{m-1}) \quad \text{in} \quad \Omega \\
w^m_{i,3} + w^m_{3,i} = g_i(w^{m-1},\eta^{m-1}), \quad q^m = w^m_{3,3} + \eta^m + g_3(w^{m-1},\eta^{m-1}) \quad \text{on} \quad \{z=0\} \\
w^m = w_0, \quad \eta^m = \eta_0 \quad \text{at} \quad t=0 \\
\eta^m_t - w^m_3 = w^{m-1}\cdot\nabla\eta^{m-1} = h(w^{m-1},\eta^{m-1}) \quad \text{on} \quad \{z=0\}, \quad w^m = 0 \quad \text{on} \quad \{z=-1\}
\end{cases}
$$

Therefore, we can apply the contraction mapping lemma to $\{w^m\}$.

### 3. SOLVABILITY of $V$

We want to prove that there exists a vector field $V$ such that $\nabla\cdot V = \sigma$ and $V=0$ at the bottom. We will look for such a $V$ in the form $V = \nabla r + \nabla\times\tau$. First, $r$ solves the following elliptic problem (E1). The existence of a solution and its regularity is well-known. For example, we can use Lax-Milgram theorem for the existence of weak solutions, Agmon-Douglis-Nirenberg theorem for the regularity. See [2].

$$
(E1)\begin{cases}
\triangle r = \sigma \quad \text{in} \quad \Omega \\
n\cdot\nabla r = 0 \quad \text{on} \quad S_B, \quad r = 0 \quad \text{on} \quad S_F
\end{cases}
$$

Then, we solve $\tau$ in terms of $r$. $\tau$ solves the following elliptic problem.

$$(E2)\begin{cases} \triangle\tau = 0 \quad \text{in} \quad \Omega \\ n\cdot\nabla\tau = -n\times\nabla r \quad \text{on} \quad S_B, \quad r=0 \quad \text{on} \quad S_F \end{cases}$$

First, $\nabla\cdot V = \sigma$ is trivial. $V=0$ at the bottom because

$$\begin{aligned} & (\partial_1 r, \partial_2 r, \partial_3 r) + (\partial_2\tau_3 - \partial_3\tau_2, \partial_3\tau_1 - \partial_1\tau_3, \partial_1\tau_2 - \partial_2\tau_1) = (\partial_1 r, \partial_2 r, \partial_3 r) + (-\partial_3\tau_2, \partial_3\tau_1, 0) \\ = \;& (\partial_1 r, \partial_2 r, 0) + (-\partial_3\tau_2, \partial_3\tau_1, 0) = 0 \end{aligned}$$

where we used the tangential boundary condition of $\tau$ for the first equality, used the Neumann boundary condition of $r$ for the second equality, and used the Neumann boundary condition of $\tau$ for the last equality. In (LCNSF), we have $V_t - \triangle V$, $V(0)$, and $V_3$.

$$V_t - \triangle V = \nabla r_t + \nabla\times\tau_t - \nabla\triangle r = \nabla r_t + \nabla\times\tau_t - \nabla\sigma$$

$\nabla r_t$ and $\nabla\times\tau_t$ are estimated by taking $\partial_t$ to (E1) and (E2) and doing the energy estimate. $V_3 = \partial_3 r \in L^2_t\dot{H}^{\frac{5}{2}}_x$ on $S_F$ by the regularity of $r$ and the trace theorem.

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