GLOBAL WELL-POSEDNESS FOR THE MICROSCOPIC FENE MODEL WITH A SHARP BOUNDARY CONDITION

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Abstract. We prove global well-posedness for the microscopic FENE model under a sharp boundary requirement. The well-posedness of the FENE model that consists of the incompressible Navier-Stokes equation and the Fokker-Planck equation has been studied intensively, mostly with the zero flux boundary condition. In this article, we show that for the well-posedness of the microscopic FENE model ($b > 2$) the least boundary requirement is that the distribution near boundary needs to approach zero faster than the distance function. Under this condition, it is shown that there exists a unique weak solution in a weighted Sobolev space. Moreover, such a condition still ensures that the distribution is a probability density. The sharpness of this boundary requirement is shown by a construction of infinitely many solutions when the distribution approaches zero no faster than the distance function.

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Contents

1. Introduction

It is well-known that the following system coupling incompressible Navier-Stokes equation for the macroscopic velocity field $v(t, x)$ and the Fokker-Planck equation

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for the probability density function $f(t,x,m)$ describes diluted solutions of polymeric liquids with noninteracting polymer chains,

$$\begin{aligned}
(1.1)\qquad \partial_t v + (v\cdot\nabla)v + \nabla p &= \nabla\cdot\tau_p + \nu\Delta v,\\
(1.2)\qquad \nabla\cdot v &= 0,\\
(1.3)\qquad \partial_t f + (v\cdot\nabla)f + \nabla_m\cdot(\nabla v m f) &= \frac{2}{\zeta}\nabla_m\cdot(\nabla_m\Psi(m)f) + \frac{2kT}{\zeta}\Delta_m f,
\end{aligned}$$

where $x\in\mathbb{R}^n$ is the macroscopic Eulerian coordinate and $m\in\mathbb{R}^n$ is the microscopic molecular configuration variable. In this model, a polymer is idealized as an elastic dumbbell consisting of two beads joined by a spring that can be modeled by a vector m (see e.g [4]). In the Navier-Stokes equation (1.1), p is hydrostatic pressure, ν is the kinematic viscosity coefficient, and τ_p is a tensor representing the polymer contribution to stress,

$$\tau_p = \lambda\int m\otimes\nabla_m\Psi(m) f\,dm,$$

where Ψ is the elastic spring potential and λ is the polymer density constant. In the Fokker-Planck equation (1.3), ζ is the friction coefficient of the dumbbell beads, T is the absolute temperature, and k is the Boltzmann constant. Notice that the Fokker-Planck equation can be written as a stochastic differential equation (see [26]).

One of the simplest model is the Hookean model in which the potential Ψ is given by

$$\Psi(m) = \frac{H|m|^2}{2},$$

where H is the elasticity constant. A more realistic model is the finite extensible nonlinear elasticity (FENE) model with

$$\Psi(m) = -\frac{Hb}{2}\log\left(1-\frac{|m|^2}{b}\right), \quad m\in B. \tag{1.4}$$

Here $B \overset{\text{def.}}{=} B(0,\sqrt{b})$ is the ball with center 0 and radius $\sqrt{b}$ which denotes the maximum dumbbell extension. In this work we shall focus our attention on the potential (1.4) and the case $b>2$, which is known to contain the parameter range of physical interest. We refer the reader to [7, 4] for a comprehensive survey of the physical background.

In past years the well-posedness of the FENE model (1.1)-(1.3) has been studied intensively in several aspects. For local well-posedness of strong solutions we refer the reader to [13] for the FENE model (in the setting where the Fokker-Planck equation is formulated by a stochastic differential equation) with $b>2$ or sometime $b>6$, [9] for a polynomial force and [28] for the FENE model with $b>76$. For a preliminary study on some related coupled PDE systems, we refer to the earlier work [27] (however, the FENE model was not addressed there). Moreover, the authors in [18] proved global existence of smooth solutions near equilibrium under some restrictions on the potential; further developments were made in subsequent works [19, 17]. More recently, N. Masmoudi [24] proved global existence for the FENE model (1.1)-(1.3) for a class of potentials (1.4) with $b>0$ assuming that the data is small, or the model is restricted to the co-rotational case in dimension two.

For results concerning the existence of weak solutions to coupled Navier-Stokes-Fokker-Planck systems and a detailed survey of related literature we refer to [1, 20, 23, 2, 3]. For an earlier result on existence of weak solutions, we refer to [8] for the Fokker-Planck equation alone with $b > 4$. On the other hand, the authors in [14], investigated the long-time behavior of both Hookean models and FENE models in several special flows in a bounded domain with suitable boundary conditions.

The complexity with the FENE potential lies mainly with the singularity of the equation at the boundary. To overcome this difficulty, several transformations relating to the equilibrium solution have been introduced in literature. See, e.g. [5, 6, 8, 21, 15]. A detailed discussion will be given in Section 2. For the microscopic FENE model, the singularity in the potential requires at least the zero Dirichlet boundary condition

$$f|_{\partial B} = 0. \tag{1.5}$$

This is consistent with the result in [12], which states that the stochastic solution trajectory does not reach the boundary almost surely. However, condition (1.5) is insufficient for wellposedness. In [21], C. Liu and H. Liu examined the ratio of the distribution f and the equilibrium f^{eq}, i.e.,

$$w = \frac{f}{f^{eq}}$$

for the microscopic FENE model, by the method of the Fichera function they were able to show that $b = 2$ is a threshold in the sense that for $b \geq 2$ any preassigned boundary value of w will become redundant, and for $b < 2$ that value has to be a priori given. As a side note we point out that there is a misprint in the statement of this result, Theorem 1.1 in [21], where the correct assertion should be about the boundary condition for w rather than for f – the proof is otherwise correct.

The boundary issue for the underlying FENE model is fundamental, and our main quest in this paper is whether one can identify a sharp boundary requirement so that both existence and uniqueness of a global weak solution to the microscopic FENE model can be established, also the solution remains a probability density. The answer is positive, and we claim that f must satisfy the following boundary condition

$$fd^{-1}|_{\partial B} = 0 \quad \text{for almost all } t > 0, \tag{1.6}$$

where $d \stackrel{\text{def.}}{=} d(m, \partial B)$ denotes the distance function from $m \in B$ to the boundary ∂B. Our claim is supported by our main results: the global well-posedness for the Fokker-Planck equation stated in Theorem 2, the property of the solution as a probability density given in Proposition 3, and the sharpness of (1.6) stated in Proposition 4.

In this article, we focus on the underlying Fokker-Planck equation (1.3) alone. Let $v(t, x)$ be the velocity field governed by (1.1) and (1.2). We assume that this underlying velocity field is smooth, then a simplification can be made by considering the microscopic model (1.3) along a particle path defined as

$$\partial_t \mathrm{X}(t, x) = v(t, \mathrm{X}(t, x)), \quad \mathrm{X}(0, x) = x.$$

For each fixed x, the distribution function $\tilde{f}(t, m; x) \stackrel{\text{def.}}{=} f(t, \mathrm{X}(t, x), m)$ solves

$$\partial_t \tilde{f} + \nabla_m \cdot (\nabla v m \tilde{f}) = \frac{2}{\zeta} \nabla_m \cdot (\nabla_m \Psi(m) \tilde{f}) + \frac{2kT}{\zeta} \Delta_m \tilde{f}.$$

By a suitable scaling ([21]), and denote $\tilde{f}$ still by $f(t,m) = \tilde{f}(t,m;x)$, we arrive at the following equation

$$\partial_t f + \nabla \cdot (\kappa m f) = \frac{1}{2}\nabla \cdot \left(\frac{bm}{\rho} f\right) + \frac{1}{2}\Delta f. \tag{1.7}$$

Here, $\rho = b - |m|^2$ and $\kappa(t) = \nabla v(t, \mathrm{X}(t,x))$ is a bounded matrix such that $\mathrm{Tr}(\kappa) = 0$. We omit m from ∇_m in (1.7) for notational convenience. In this paper we prove well-posedness of (1.7) subject to some side conditions. The well-posedness of the full coupled system (1.1)-(1.3) is the subject of a forthcoming paper [22].

A weak solution of the Fokker-Planck equation (1.7) with the initial condition

$$f(0,m) = f_0(m), \quad m \in B, \tag{1.8}$$

and boundary requirement (1.6) is defined in the following.

Definition 1. *We say f is a weak solution of* (1.7), (1.8), *and* (1.6) *if the following conditions are satisfied:*

For an arbitrary subdomain B' of B such that $\overline{B'} \subset B$ and almost all $t \in (0,T)$,

(1) $f \in L^2(0,T;H^1(B'))$ *and* $\partial_t f \in L^2(0,T;H^{-1}(B'))$,

(2) *for any* $\varphi \in C_c^1(B)$,

$$\int_B \left[\partial_t f \varphi - f\kappa m \cdot \nabla\varphi + \frac{bfm\cdot\nabla\varphi}{2\rho} + \frac{1}{2}\nabla f \cdot \nabla \varphi\right] dm = 0, \tag{1.9}$$

(3)

$$f(0,m) = f_0(m) \quad \text{in } L^2(B'), \tag{1.10}$$

(4) *and for* $B_r \overset{def.}{=} B(0,r)$,

$$\lim_{r\to\sqrt{b}} ||fd^{-1}\,|_{\partial B_r}\,||_{L^2(\partial B_r)} = 0. \tag{1.11}$$

Note that (1.10) makes sense since $f \in C([0,T];L^2(B'))$ implied by (1) above, and also $fd^{-1}\,|_{\partial B_r}$ is well defined in $L^2(\partial B_r)$ by the standard trace theorem.

Regarding the weak solution defined above, several remarks are in order.

- The reason for taking compactly supported functions as test functions in Definition 1 is that we want to avoid any priori restriction to a particular weighted Sobolev space. It is this treatment that allows us to prove sharpness of boundary condition (1.11).
- Boundary condition (1.11) ensures that $f(t,\cdot) \in L^1(B)$ for each t. Indeed, we can choose $r_0 \in (0,\sqrt{b})$ such that if $r \geq r_0$

  $$||fd^{-1}|_{\partial B_r}||_{L^2(\partial B_r)} \leq 1.$$

  Then

  $$\begin{aligned}\int_B |f|dm &= \int_{B_{r_0}} |f|dm + \int_{r_0}^{\sqrt{b}}\int_{\partial B_r} |f|dSdr \\ &\leq C_1||f||_{L^2(B_{r_0})} + C_2\int_{r_0}^{\sqrt{b}} ||fd^{-1}|_{\partial B_r}||_{L^2(\partial B_r)}dr < \infty.\end{aligned}$$

- Boundary condition (1.11) or (1.6) is, in its type, different from the zero flux boundary condition

$$\left(\frac{bmf}{2\rho}+\frac{1}{2}\nabla f-\kappa mf\right)\cdot\frac{m}{|m|}\Big|_{\partial B}=0, \tag{1.12}$$

which is known to preserve the conservation property, and has been used in many priori works. The relation of these two types of boundary conditions will be discussed in Section 2 as well.

In order to establish an existence theorem, we now identify a subspace of $H^1(B)$ with an appropriate weight to incorporate boundary requirement (1.11). For simplicity, we consider the case with trivial velocity field such that $\kappa=0$, then equation (1.7) becomes

$$\partial_t f=\frac{1}{2}\nabla\cdot\left(\rho^{b/2}\nabla\left(\frac{f}{\rho^{b/2}}\right)\right).$$

It follows from this conservative form that the only equilibrium solution f^{eq} must be a multiplier of $\rho^{b/2}$, i.e.

$$f^{eq}=Z^{-1}\rho^{b/2},$$

where Z is a normalization factor such that $\int_B f^{eq}dm=1$.

We are interested in the case

$$b>2. \tag{1.13}$$

In such a case f^{eq} satisfies boundary requirement (1.6). Moreover

$$f^{eq}\in H^1_{-b/2}(B).$$

Here, $H^1_{-b/2}(B)=\{\phi:\phi,\partial_{m_j}\phi\in L^2_{-b/2}(B)\}$ with

$$L^2_{-b/2}(B)=\left\{\phi:\int_B\phi^2(b-|m|^2)^{-b/2}dm<\infty\right\}.$$

Our main results are summarized in Theorem 2, Proposition 3 and Proposition 4 below.

Theorem 2. *Assume* (1.13) *and* $\kappa(t)\in C[0,T]$ *for a given* $T>0$.
(i) If

$$f_0(m)\in L^2_{-b/2}(B), \tag{1.14}$$

then there exists a unique solution f *of* (1.7), (1.8), *and* (1.6) *in the sense of Definition 1. Moreover,*

$$\max_{0\le t\le T}||f(t,\cdot)||_{L^2_{-b/2}(B)}+||f||_{L^2(0,T;H^1_{-b/2}(B))}+||\partial_t f||_{L^2(0,T;((H^1{}_{-b/2})^*(B))}\le C||f_0||_{L^2_{-b/2}(B)}. \tag{1.15}$$

(ii) For any

$$f_0(m)\in L^2_{loc}(B),$$

there exists at most one solution f .

Proof. The proof of (i) will be given in Section 3 - 5. In order to prove (ii) we assume that f_1,f_2 are two weak solutions of the problem with arbitrary initial data $f_0(m)$. Then f_1-f_2 solves (1.7) with zero initial data which is in $L^2_{-b/2}$. From (1.15) in (i) it follows that $f_1\equiv f_2$ in $L^2(0,T;H^1_{-b/2})$. □

We remark that the restriction on b in (1.13) is essential to obtain the energy estimate (1.15).

The weak solution thus obtained is indeed a probability density. More precesely we have the following.

Proposition 3. *Let f be a weak solution to* (1.7)*,* (1.8)*, and* (1.6) *defined in Definition 1 subject to condition* (1.14)*. Then,*

$$\int_B f(t,m)dm = \int_B f_0(m)dm, \quad \forall t > 0. \tag{1.16}$$

Furthermore if $f_0(m) \geq 0$ a.e. on B, then $f(t,\cdot) \geq 0$ a.e. on B for all $t > 0$.

This proposition will be proved in Section 2.

The following proposition states that boundary condition (1.6) is sharp for the uniqueness of the weak solution.

Proposition 4. *Assume* (1.13) *and $\kappa(t) \in C[0,T]$. If boundary condition* (1.6) *fails, that is,*

$$fd^{-1}|_{\partial B} \neq 0$$

is assumed, then the Fokker-Planck equation (1.7) *with $f_0(m) = 0$ has infinitely many solutions.*

In other words, Proposition 4 implies that part (ii) in Theorem 2 would fail if boundary requirement (1.6) were weaken so that near boundary the distribution approaches zero no faster than the distance function.

The justification of sharpness follows from the existence of a Cauchy-Dirichlet problem for w defined by

$$w = \frac{f}{\rho} - g \tag{1.17}$$

with g being a class of functions properly constructed.

This article is organized as follows. In Section 2, we prove Proposition 3 and provide some preliminaries including: (1) several transformations used to handle the boundary difficulty, (2) equivalence of two weighted function spaces, and (3) the relation of our boundary condition to the natural flux boundary condition. In Section 3, we transform the Fokker-Planck equation to certain Cauchy-Dirichlet problem, named as W-problem, and define a weak solution of W-problem in a weighted Sobolev space. The well-posedness of the W-problem is shown in Section 4 by the Galerkin method and the Banach fixed point theorem. This leads to the well-posedness of the Fokker-Planck equation, Theorem 2; details of the proof are presented in Section 5. In Section 6, we construct non-trivial solutions for the Fokker-Planck equation described in Proposition 4.

2. Preliminaries

2.1. **Probability density.** With the definition of our weak solution given in Definition 1 we shall show that f has the usual properties of a probability density function (i.e. it is non-negative and has a unit integral over B for all $t > 0$ if it is so initially) – this is to prove Proposition 3.

Given f_0 in $L^2_{-b/2}(B)$ and $f_0 \geq 0$ a.e. on B, we define $f_{0,l} = \eta_l * f_0 \in C^\infty(B)$ for $l \geq 1$. Here $\eta_l(m) = l^n \eta(lm)$ denotes the usual scaled mollifier. We have

$$\lim_{l\to\infty} \|f_{0,l} - f_0\|_{L^2_{-b/2}(B)} = 0, \quad f_{0,l} \geq 0.$$

Suppose that f_l is the weak solution of (1.7), (1.8), and (1.6) subject to initial condition $f_l(0, m) = f_{0,l}(m) \in C^\infty(B)$. Then, for any $T > 0$ and $0 < t < T$,

$$\left|\int_B (f(t,m) - f_l(t,m))dm\right| \leq C \max_{0\leq t\leq T} \|f(t,\cdot) - f_l(t,\cdot)\|_{L^2_{-b/2}(B)} \leq C\|f_0 - f_{0,l}\|_{L^2_{-b/2}(B)}. \tag{2.1}$$

Hence for justification of the conservation of polymers, it suffices to prove that

$$\int_B f_l(t,m)dm = \int_B f_l(0,m)dm, \quad \forall t \geq 0. \tag{2.2}$$

To do so, we take a test function $\varphi_\varepsilon \in C_c^\infty(B)$ converging to χ_B as $\varepsilon \to 0$ such that

$$\varphi_\varepsilon(m) = \begin{cases} 1, & |m| \leq \sqrt{b} - \varepsilon \\ 0, & |m| \geq \sqrt{b} - \varepsilon/2 \end{cases}$$

and

$$|\partial_{m_i}\varphi_\varepsilon| < C\frac{1}{\varepsilon}, \quad |\partial_{m_i}\partial_{m_j}\varphi_\varepsilon| < C\frac{1}{\varepsilon^2}. \tag{2.3}$$

From (1.9) and the fact that derivatives of φ_ε are supported in $B^\varepsilon := B_{\sqrt{b}-\varepsilon/2} \backslash B_{\sqrt{b}-\varepsilon}$, we have

$$\int_B \partial_t f_l \varphi_\varepsilon dm = \int_{B^\varepsilon} \left[f_l \kappa m \cdot \nabla\varphi_\varepsilon - \frac{b f_l m \cdot \nabla\varphi_\varepsilon}{2\rho} - \frac{1}{2}\nabla f_l \cdot \nabla\varphi_\varepsilon\right] dm. \tag{2.4}$$

Applying the mean value theorem of the form

$$\int_{B^\varepsilon} g dm = \frac{\varepsilon}{2}\int_{\partial B_r} g dS, \quad \text{for some } r \in (\sqrt{b} - \varepsilon, \sqrt{b} - \varepsilon/2),$$

to the first term on the right of (2.4) together with (2.3), we obtain

$$\left|\frac{\varepsilon}{2}\int_{\partial B_r} f_l \kappa m \cdot \nabla\varphi_\varepsilon dS\right| \leq C\int_{\partial B_r} |f_l| dS \leq C\|f_l d^{-1}\|_{L^2(\partial B_r)}.$$

Similarly the second term on the right of (2.4) is bounded by

$$\int_{\partial B_r} \left|\frac{f_l}{\rho}\right| dS \leq C\|f_l d^{-1}\|_{L^2(\partial B_r)}.$$

It follows from (1.11) that the above two upper bounds converge to zero as $\varepsilon \to 0$.

Integration by parts in the last term in (2.4) yields

$$\begin{aligned}\left|\int_{B^\varepsilon} \nabla f_l \cdot \nabla\varphi_\varepsilon dm\right| &\leq \int_{B^\varepsilon} |f_l \Delta\varphi_\varepsilon| dm + \int_{\partial B^\varepsilon} |f_l \nabla\varphi_\varepsilon| dS \\ &= \frac{\varepsilon}{2}\int_{\partial B_r} |f_l \Delta\varphi_\varepsilon| dS + \int_{\partial B^\varepsilon} |f_l \nabla\varphi_\varepsilon| dS \\ &\leq C\int_{\partial B_r \cup \partial B^\varepsilon} \frac{|f_l|}{\varepsilon} dS,\end{aligned}$$

which, in virtue of $|f_l|/\varepsilon \le |f_l|d^{-1}$ on $\partial B_r \cup \partial B^\varepsilon$, is converging to zero as $\varepsilon \to 0$ as well.

Due to Theorem 2 and $f_{0,l} \in C^\infty(B)$, it follows that $\partial_t f_l$ is bounded in any B_r for $0 < r < \sqrt{b}$. Thus, for any $\tau, s > 0$

$$\begin{aligned} \left| \int_B f_l(\tau, m)\varphi_\varepsilon dm - \int_B f_l(s, m)\varphi_\varepsilon dm \right| &= \left| \int_s^\tau \frac{d}{dt} \left(\int_B f_l(t, m)\varphi_\varepsilon dm \right) dt \right| \\ &= \left| \int_s^\tau \int_B \partial_t f_l(t, m)\varphi_\varepsilon dm dt \right|. \end{aligned}$$

Using the estimate for $\int_B \partial_t f_l(t, m)\varphi_\varepsilon dm$ together with the boundedness of $f_l(t, m)$, we can send ε to zero to obtain (2.2) as claimed.

We now turn to justify the positivity. Consider the transformation introduced in [21]

$$f_l = w_l \rho^{b/2-\alpha} e^{Kt}. \tag{2.5}$$

Then w_l solves

$$\rho^2 \partial_t w_l - \frac{1}{2}\rho^2 \Delta w_l - \rho[\frac{4\alpha - b}{2} m - \rho\kappa m] \cdot \nabla w_l - c(m) w_l = 0, \tag{2.6}$$

where

$$c(m) = -K\rho^2 + \alpha[nb + (2\alpha + 2 - n - b)|m|^2] + (b - 2\alpha)\rho m \cdot \kappa m.$$

Then for any $\overline{B_r} \subset B$, w_l is a classical solution in $(0, T] \times B_r$. It was shown in [21] that there exist $\alpha < b/2 - 1$ and K so that $c(m) < 0$. The maximum principle yields that w_l can not achieve a negative minimum at the interior points of $[0, T] \times B_r$. Thus the negative minimum of w_l, if it exists, can only be attained on the parabolic boundary of the domain.

From the transformation (2.5) and the condition $b/2 - \alpha > 1$, it follows that the negative minimum of f_l, if any, can only be attained at the initial time. Therefore

$$f_l \ge -\max f_{0,l}^- \ge 0.$$

Now fix t. For any x_0 and η such that $B(x_0; \eta) \subset B$,

$$-\int_{B(x_0;\eta)} f dm + \int_{B(x_0;\eta)} f_l dm \le \int_{B(x_0;\eta)} |f - f_l| dm \le \int_B |f - f_l| dm \le C||f_0 - f_{0,l}||_{L^2_{-b/2}}.$$

Here (2.2) has been used to obtain the last inequality. Hence

$$\int_{B(x_0;\eta)} f dm \ge \int_{B(x_0;\eta)} f_l dm - C||f_{0,l} - f_0||_{L^2_{-b/2}},$$

which as $l \to \infty$ leads to

$$\int_{B(x_0,\eta)} f \ge 0.$$

Since x_0 and η are arbitrary, $f \ge 0$ almost everywhere on B for $t > 0$. The proof of Proposition 3 is now complete.

2.2. **Transformations.** To overcome the difficulty caused by the boundary singularity, several transformations have been introduced in literature. With boundary condition (1.6), in this work we introduce

$$w = \frac{f}{\rho}$$

to transform the Fokker-Planck equation to a degenerate parabolic equation with zero boundary condition (see details in Section 3). A widely accepted transformation is the ratio of the unknown to the equilibrium solution, i.e.,

$$w = \frac{f}{\rho^{b/2}}.$$

Such a transformation was used in [21] to reformulate the Fokker-Planck equation, and examine whether a Dirichlet type boundary condition is necessary.

A third transformation is

$$w = \frac{f}{\rho^{b/4}}.$$

This was used in [8, 11] to remove the singularity at the boundary in the resulting equation. It was also used in [15] to formulate a weak formulation of w for discretization using a spectral Galerkin approximation.

Another transformation defined by

$$w = \frac{f}{\rho^{s}}$$

with $b \geq 4s^2/(2s-1)$ and $s > 1/2$ may also lead to a well-posed problem. The minimum value of the function $4s^2/(2s-1)$ is attained at $s = 1$, yielding the maximum range of b values, $b \geq 4$. This transformation was proposed in [5, 6] in the special case $s = 2$ and $s = 2.5$, where these values were chosen on the basis of numerical experiments in two and three dimensions, respectively. We note that our transformation $w = f/\rho$ corresponds to $s = 1$, but not limited by $b \geq 4$.

2.3. **Weighted Sobolev spaces.** In Section 1, we defined the weighted Sobolev space $H^1_{-b/2}(B)$. For a more general nonnegative measurable weight function σ, a weighted Sobolev space $H^1(\Omega;\sigma)$ is defined as a set of measurable function ϕ such that

$$||\phi||^2_{H^1(\Omega;\sigma)} := \int_\Omega (|\nabla\phi|^2 + \phi^2)\sigma dm < \infty.$$

Similarly, a weighted $L^2(\Omega;\sigma)$ can be defined. $\overset{\circ}{H}{}^1(\Omega;\sigma)$ denotes a completion of $C_c^\infty(\Omega)$ with $||\cdot||_{H^1(\Omega;\sigma)}$. It is obvious that $H^1(\Omega;\sigma)$ and $\overset{\circ}{H}{}^1(\Omega;\sigma)$ are Hilbert spaces with the inner product $\langle\cdot,\cdot\rangle_{H^1(\Omega;\sigma)}$ defined as

$$\langle\phi_1,\phi_2\rangle_{H^1(\Omega;\sigma)} = \int_\Omega (\nabla\phi_1\cdot\nabla\phi_2 + \phi_1\phi_2)\sigma dm$$

and

$$\overset{\circ}{H}{}^1(\Omega;\sigma) \subset H^1(\Omega;\sigma).$$

For notational convenient, we use $H^1_\mu(\Omega)$, $\overset{\circ}{H}{}^1_\mu(\Omega)$ and $L^2_\mu(\Omega)$ for $H^1(\Omega;\rho^\mu)$, $\overset{\circ}{H}{}^1(\Omega;\rho^\mu)$ and $L^2(\Omega;\rho^\mu)$ respectively. We also omit the domain Ω if it is obvious.

In contrast to the standard weighted Sobolev space $H^1_{-b/2}(B)$ used in this work, the following weighted function space

$$\rho^{b/2}H^1_{b/2}(B) := \left\{\phi : \quad \frac{\phi}{\rho^{b/2}} \in H^1_{b/2}(B)\right\}$$

is well known in literature for Fokker-Planck equations with FENE potentials, see e.g. [1, 14, 2, 20, 24, 15]. We now show their equivalence as long as $b > 2$.

The key estimate we need to prove the equivalence is the embedding theorem stated in Lemma 5. Set $\psi = \phi\rho^{-b/2}$. If $\phi \in H^1_{-b/2}(B)$, we use the relation

$$\nabla\psi = \frac{\nabla\phi}{\rho^{b/2}} - \frac{2m}{\rho^{b/2+1}}\phi.$$

It is obvious that

$$\frac{\nabla\phi}{\rho^{b/2}} \in L^2_{b/2}(B).$$

Also the use of Lemma 5 and the fact that $H^1_{-b/2}(B) = \mathring{H}^1_{-b/2}(B)$ for $b > 2$ (see [16]) give

$$\left\|\frac{\phi}{\rho^{b/2+1}}\right\|_{L^2_{b/2}} = \|\phi\|_{L^2_{-b/2-2}} \le C\|\phi\|_{H^1_{-b/2}}.$$

Hence $\phi \in \rho^{b/2}H^1_{b/2}(B)$. If $\phi \in \rho^{b/2}H^1_{b/2}(B)$ we use the following identity

$$\nabla\phi = \rho^{b/2}\nabla\psi - bm\rho^{b/2-1}\psi.$$

It is easy to see that $\rho^{b/2}\nabla\psi \in L^2_{-b/2}$; also for $b > 2$ we have

$$\|\rho^{b/2-1}\psi\|_{L^2_{-b/2}} = \|\psi\|_{L^2_{b/2-2}} \le C\|\psi\|_{H^1_{b/2}}$$

by Lemma 5. Thus $\phi \in H^1_{-b/2}(B)$. These together verify that $\rho^{b/2}H^1_{b/2}$ and $H^1_{-b/2}$ are equivalent when $b > 2$.

2.4. **Boundary conditions.** Granted certain smoothness of f, e.g. $f \in C^1(\bar{B})$, one may argue that our boundary condition (1.6) is equivalent to the zero flux boundary condition (1.12).

Set $\nu = \frac{m}{|m|}$ and $g = fd^{-1}$. We calculate the flux

$$\begin{aligned} J &:= \left(\frac{bmf}{2\rho} - \kappa mf + \frac{1}{2}\nabla f\right) \cdot \frac{m}{|m|} \\ &= \frac{b|m|}{2(|m| + \sqrt{b})}\frac{f}{d} + \frac{1}{2}\frac{\partial f}{\partial\nu} - |m|\nu \cdot \kappa\nu f. \end{aligned}$$

Due to singularity on boundary it is necessary that $f|_{\partial B} = 0$. For any point $p \in \partial B$, let m be a point in B such that $m + d\nu = p$. Then

$$\frac{\partial f}{\partial\nu}(p) = -\lim_{d\to 0}\frac{f(m)}{d}.$$

We thus have

$$J(p) = \lim_{d\to 0} J(m) = \frac{1}{4}(b-2)\lim_{d\to 0}\frac{f(m)}{d}.$$

This implies that $J(p) = 0$ if and only if

$$\lim_{d\to 0} \frac{f(m)}{d} = 0.$$

3. Transformation of the microscopic FENE model

In what follows we shall call the Fokker-Planck equation (1.7) with initial condition (1.8) and boundary condition (1.6) as the Fokker-Planck-FENE (FPF) problem. We first formulate a time evolution equation from the FPF problem. Define $w(t, m)$ as

$$f(t, m) = w(t, m)\rho. \tag{3.1}$$

Then (1.7) is transformed to

$$\rho \left[\partial_t w - \frac{1}{2}\Delta w - \frac{(b-4)m - 2\rho\kappa m}{2\rho} \cdot \nabla w - \frac{c}{\rho} w \right] = 0, \tag{3.2}$$

where

$$c(t, m) = 2m \cdot \kappa(t)m + n(b/2 - 1). \tag{3.3}$$

Setting a parameter

$$\beta = -\frac{b}{2} + 2,$$

we rewrite (3.2) as

$$\rho^{b/2-1} \left[\partial_t w\rho^\beta - \frac{1}{2}\nabla \cdot (\nabla w\rho^\beta) + \kappa m \cdot \nabla w\rho^\beta - cw\rho^{\beta-1} \right] = 0.$$

Boundary condition (1.6) implies that $w(t, \cdot)$ satisfies a homogeneous boundary condition for almost all t since the distance function d and ρ are equivalent (see (3.9)).

The FPF problem is formally transformed to the following W-problem:

$$\partial_t w\rho^\beta - \frac{1}{2}\nabla \cdot (\nabla w\rho^\beta) + \kappa m \cdot \nabla w\rho^\beta - cw\rho^{\beta-1} = 0, \qquad (t, m) \in (0, T] \times B, \tag{3.4}$$

$$w(0, m) = w_0(m), \qquad m \in B, \tag{3.5}$$

$$w(t, m) = 0, \qquad (t, m) \in [0, T] \times \partial B. \tag{3.6}$$

Here,

$$w_0(m) = f_0(m)\rho^{-1}$$

according to the transformation (3.1).

In order to define a weak solution of this W-problem, we first state a useful lemma.

Lemma 5. *Suppose that* $\Omega = B$.

(1) *If* $\phi \in \overset{\circ}{H}{}^1_\mu$ *for* $\mu < 1$, *then*

$$||\phi||_{L^2_{\mu-2}} \le C_0 ||\phi||_{H^1_\mu}. \tag{3.7}$$

If $\mu > 1$, *we have the same inequality for* $\phi \in H^1_\mu$

(2) *If* $\phi \in H^1_\mu$ *for* $\mu < 1$, *then the trace map*

$$\begin{aligned} \mathcal{T}: \quad H^1_\mu(\Omega) &\to L^2(\partial\Omega) \\ \phi \quad &\mapsto \phi|_{\partial\Omega} \end{aligned}$$

is well defined, i.e. it is a bounded linear map.

In particular, for $\phi \in \overset{\circ}{H}{}^1_\mu$

$$\mathcal{T}(\phi) = 0. \tag{3.8}$$

Proof. In [25](see also [16]), it was proved that

$$\overset{\circ}{H}{}^1(\Omega; d^\mu) \hookrightarrow L^2(\Omega; d^{\mu-2})$$

provided $\partial\Omega$ is Lipschitz continuous. Recall that d denotes the distance from m to the boundary of Ω. (3.7) follows from

$$\sqrt{b}d \leq \rho \leq 2\sqrt{b}d. \tag{3.9}$$

It is also known that the trace map $\mathcal{T}$ is well defined for $0 \leq \mu < 1$ ([25, 16]). For $\mu < 0$

$$||\phi||_{H^1} \leq b^{-\mu/2}||\phi||_{H^1_\mu}$$

since $\rho^\mu \geq b^\mu$ for all $m \in B$. Therefore, $\mathcal{T}$ is well defined for $\mu < 1$. (3.8) is obvious from the definitions of the trace map and $\overset{\circ}{H}{}^1_\mu$. □

We now define a weak solution to W-problem in a standard manner. Multiplication by a test function $\varphi \in C^1_c(B)$ to the equation (3.4) and integration over B yield

$$\int_B \left[\partial_t w \varphi \rho^\beta + \frac{1}{2}\nabla w \cdot \nabla \varphi \rho^\beta + \kappa m \cdot \nabla w \varphi \rho^\beta - c w \varphi \rho^{\beta-1}\right] dm = 0.$$

This equation is well defined assuming that $\partial_t w(t,\cdot) \in (\overset{\circ}{H}{}^1_\beta)^*$, the dual space of $\overset{\circ}{H}{}^1_\beta$, and $w(t,\cdot), \varphi \in \overset{\circ}{H}{}^1_\beta$ due to the boundedness of c and Lemma 5. Moreover,

$$\overset{\circ}{H}{}^1_\beta \subset L^2_\beta$$

implies

$$\overset{\circ}{H}{}^1_\beta \subset L^2_\beta \subset (\overset{\circ}{H}{}^1_\beta)^*.$$

Thus

$$w(t,x) \in C([0,T]; L^2_\beta).$$

Here we identify L^2_β with its dual space.

Let $(\cdot,\cdot)_H$ denote the paring of a Hilbert space H with its dual space H^* and

$$\mathbf{L}[w,\varphi;t] = \frac{1}{2}\int_B \nabla w(t,m) \cdot \nabla \varphi \rho^\beta dm + \int_B \kappa m \cdot \nabla w(t,m) \varphi \rho^\beta dm. \tag{3.10}$$

We now describe the weak solution we are looking for.

Definition 6. *A function* $w(t,m)$ *such that*

$$w(t,m) \in L^2(0,T; \overset{\circ}{H}{}^1_\beta), \text{ with } \partial_t w(t,m) \in L^2(0,T; (\overset{\circ}{H}{}^1_\beta)^*)$$

is a weak solution of W*-problem,* (3.4)-(3.6)*, provided*

(1) *For each* $\varphi \in \overset{\circ}{H}{}^1_\beta$ *and almost every* $0 \le t \le T$,

$$(\partial_t w(t,\cdot), \varphi)_{\overset{\circ}{H}{}^1_\beta} + \boldsymbol{L}[w, \varphi; t] = \int_B c w(t,m)\varphi\rho^{\beta-1} dm,$$

(2) $w(0,m) = w_0(m)$ *in* L^2_β *sense. i.e.*

$$\int_B |w(0,m) - w_0(m)|^2 \rho^\beta dm = 0.$$

The following energy estimate for $\mathbf{L}[w, w; t]$ for fixed t is crucial.

Lemma 7. *There exist positive constants* C_1 *and* C_2 *depending only on* b *and* $||\kappa||_{L^\infty(0,T)}$ *such that*

$$C_1 ||w(t,\cdot)||^2_{H^1_\beta} \le \boldsymbol{L}[w, w; t] + C_2 ||w(t,\cdot)||^2_{L^2_\beta}.$$

Proof. Let $\phi = w$ in (3.10) and apply the Schwarz inequality we arrive at the above estimate as desired. □

The well-posedness of the W-problem is stated in the following

Lemma 8. W*-problem,* (3.4)-(3.6)*, is uniquely solvable in weak sense for* $w_0 \in L^2_\beta$*. Furthermore,*

$$\max_{0\le t\le T} ||w(t,\cdot)||_{L^2_\beta} + ||w||_{L^2(0,T;\overset{\circ}{H}{}^1_\beta)} + ||\partial_t w||_{L^2(0,T;(\overset{\circ}{H}{}^1_\beta)^*)} \le C||w_0||_{L^2_\beta}.$$

A detailed proof of this result will be presented in next section.

4. Well-posedness for the transformed problem

In this section, we show the well-posedness of the weak solution to W-problem. For this aim, we consider the following U-problem containing a non-homogeneous term $h(t,m) \in L^2(0,T; L^2_{2-\beta})$.

$$\partial_t u\rho^\beta - \frac{1}{2}\nabla\cdot(\nabla u\rho^\beta) + \kappa m\cdot\nabla u\rho^\beta - h = 0, \qquad (t,m) \in (0,T] \times B, \tag{4.1}$$

$$u(0,m) = u_0(m), \qquad m \in B, \tag{4.2}$$

$$u(t,m) = 0, \qquad (t,m) \in [0,T] \times \partial B. \tag{4.3}$$

The weak solution of U-problem is defined similarly.

Definition 9. *We say a function* u *such that*

$$u \in L^2(0,T; \overset{\circ}{H}{}^1_\beta), \text{ with } \partial_t u \in L^2(0,T; (\overset{\circ}{H}{}^1_\beta)^*)$$

is a weak solution of U*-problem provided*

(1) *for each* $\varphi \in \overset{\circ}{H}{}^1_\beta$ *and almost every* $0 \le t \le T$

$$(\partial_t u(t,\cdot), \varphi)_{\overset{\circ}{H}{}^1_\beta} + \mathbf{L}[u, \varphi; t] = \int_B h(t,m)\varphi dm,$$

(2) $u(0,m) = u_0(m)$ *in* L^2_β.

We note that $\int_B h(t,m)\varphi dm$ is finite for any $h(t,\cdot) \in L^2_{2-\beta}$ since $\varphi \in L^2_{\beta-2}$ from (3.7). Thus $\int_B h(t,m)\varphi dm$ can be understood as the L^2_0 inner product although $h(t,\cdot)$ may not belong to L^2_0.

The well-posedness for U-problem follows from the standard Galerkin method.

Lemma 10. *For given $h \in L^2(0,T;L^2_{2-\beta})$ and $u_0 \in L^2_\beta$, U-problem has a unique weak solution. Moreover,*

$$\max_{0\le t\le T} ||u(t,\cdot)||_{L^2_\beta} + ||u||_{L^2(0,T;\mathring{H}^1_\beta)} + ||\partial_t u||_{L^2(0,T;(\mathring{H}^1_\beta)^*)} \le C(||h||_{L^2(0,T;L^2_{2-\beta})} + ||u_0||_{L^2_\beta}). \tag{4.4}$$

Proof. We first construct an approximate solution in a finite-dimensional space. Let $\{\phi_i\}$ be a basis of $\mathring{H}^1_\beta$ and L^2_β. The existence of such a basis can be verified from the fact that $\mathring{H}^1_\beta$ is a dense subset of L^2_β. Consider an approximation $u_l(t,m) = \sum_{i=1}^{l} d_i^l(t)\phi_i$, where d_i^l satisfies

$$(\partial_t u_l(t,\cdot), \phi_j)_{\mathring{H}^1_\beta} + \mathbf{L}[u_l, \phi_j; t] = \langle h(t,\cdot), \phi_j\rangle_{L^2_0}, \quad 1 \le j \le l, \tag{4.5}$$

$$\sum_{i=1}^{l} d_i^l(0)\phi_i \to u_0 \text{ in } L^2_\beta, \text{ as } l \to \infty. \tag{4.6}$$

Since (4.5) and (4.6) form a system of linear differential equations by the best approximation theory, $\{d_i^l\}$ is uniquely determined for each l. We rewrite (4.5) as

$$\langle \partial_t u_l(t,\cdot), \phi_j\rangle_{L^2_\beta} + \mathbf{L}[u_l, \phi_j; t] = \langle h(t,\cdot), \phi_j\rangle_{L^2_0}, \quad 1 \le j \le l. \tag{4.7}$$

Apply d_j^l to (4.7) and sum for $1 \le j \le l$, then for almost every t

$$\langle \partial_t u_l(t,\cdot), u_l(t,\cdot)\rangle_{L^2_\beta} + \mathbf{L}[u_l, u_l; t] = \langle h(t,\cdot), u_l(t,\cdot)\rangle_{L^2_0}.$$

From Lemma 7, it follows that

$$\frac{d}{dt}||u_l(t,\cdot)||^2_{L^2_\beta} + 2C_1||u_l(t,\cdot)||^2_{H^1_\beta} \le 2C_2||u_l(t,\cdot)||^2_{L^2_\beta} + 2\langle h(t,\cdot), u_l(t,\cdot)\rangle_{L^2_0}. \tag{4.8}$$

From (3.7), for any $\delta > 0$

$$|\langle h(t,\cdot), u_l(t,\cdot)\rangle_{L^2_0}| \le \frac{1}{2\delta}||h(t,\cdot)||^2_{L^2_{2-\beta}} + \frac{\delta}{2}C_0^2||u_l(t,\cdot)||^2_{H^1_\beta}.$$

With $\delta = C_1/C_0^2$, (4.8) can be rewritten as

$$\frac{d}{dt}||u_l(t,\cdot)||^2_{L^2_\beta} + C_1||u_l(t,\cdot)||^2_{H^1_\beta} \le 2C_2||u_l(t,\cdot)||^2_{L^2_\beta} + C_0^2/C_1||h(t,\cdot)||^2_{L^2_{2-\beta}}, \tag{4.9}$$

or

$$\frac{d}{dt}||u_l(t,\cdot)||^2_{L^2_\beta} \le 2C_2||u_l(t,\cdot)||^2_{L^2_\beta} + C_0^2/C_1||h(t,\cdot)||^2_{L^2_{2-\beta}}.$$

Use Gronwall's inequality to obtain

$$\max_{0\le t\le T} ||u_l(t,\cdot)||^2_{L^2_\beta} \le C\left(||u_0||^2_{L^2_\beta} + ||h||^2_{L^2(0,T;L^2_{2-\beta})}\right),$$

where C is an appropriate constant which may depend on β, b, T and $|\kappa|$. On the other hand, integration of (4.9) from 0 to T together with above inequality yields

$$||u_l||^2_{L^2(0,T;\overset{\circ}{H}{}^1_\beta)} \leq C\left(||u_0||^2_{L^2_\beta} + ||h||^2_{L^2(0,T;L^2_{2-\beta})}\right). \tag{4.10}$$

A similar argument to that in [10] gives us the estimate for $||\partial_t u_l||$ as

$$||\partial_t u_l||^2_{L^2(0,T;(\overset{\circ}{H}{}^1_\beta)^*)} \leq C\left(||u_0||^2_{L^2_\beta} + ||h||^2_{L^2(0,T;L^2_{2-\beta})}\right).$$

Here we have used (4.5) with $\phi \in \overset{\circ}{H}{}^1_\beta$ such that $||\phi||_{H^1_\beta} \leq 1$ and (4.10). By passing to the limit as $l \to \infty$ and a standard argument (e.g. see [10]), we have well-posedness for U-problem. □

Now, we introduce a linear map $\mathcal{A}$ to connect W and U-problems as

$$\begin{aligned}\mathcal{A}:\ \ L^2(0,\tau;L^2_\beta) &\to L^2(0,\tau;L^2_{2-\beta})\\ w &\mapsto cw\rho^{\beta-1}.\end{aligned}$$

Since c is bounded,

$$\begin{aligned}||cw(t,\cdot)\rho^{\beta-1}||^2_{L^2_{2-\beta}} &\leq ||c||^2_{L^\infty}\int_B w^2(t,\cdot)\rho^{2\beta-2}\rho^{2-\beta}dm,\\ &= ||c||^2_{L^\infty}\int_B w^2(t,\cdot)\rho^{\beta}dm.\end{aligned}$$

Thus, $\mathcal{A}$ is well defined and

$$||\mathcal{A}(w_1)-\mathcal{A}(w_2)||^2_{L^2(0,T;L^2_{2-\beta})} \leq ||c||^2_{L^\infty}||w_1-w_2||^2_{L^2(0,T;L^2_\beta)}.$$

We define another map $\mathcal{F}$ such that

$$\begin{aligned}\mathcal{F}:\ \ C([0,\tau];L^2_\beta) &\to C([0,\tau];L^2_\beta)\\ w &\mapsto u.\end{aligned}$$

Here, $\mathcal{F}(w)$ is given by the weak solution of U-problem with

$$h = \mathcal{A}(w),$$

and the initial condition

$$u_0(m) = w(0,m).$$

The map $\mathcal{F}$ is well defined from Lemma 10 and the definition of $\mathcal{A}$. Now we show that $\mathcal{F}$ is a contraction mapping for sufficiently small τ. Let

$$u_1 = \mathcal{F}(w_1), \quad u_2 = \mathcal{F}(w_2).$$

From the energy estimate (4.4),

$$\begin{aligned}||u_1-u_2||^2_{C([0,\tau];L^2_\beta)} &\leq C||\mathcal{A}(w_1)-\mathcal{A}(w_2)||^2_{L^2(0,\tau;L^2_{2-\beta})}\\ &= C\int_0^\tau ||\mathcal{A}(w_1)(t,\cdot)-\mathcal{A}(w_2)(t,\cdot)||^2_{L^2_{2-\beta}}dt\\ &\leq C\int_0^\tau ||w_1(t,\cdot)-w_2(t,\cdot)||^2_{L^2_\beta}dt\\ &\leq C\tau||w_1-w_2||^2_{C([0,\tau];L^2_\beta)}.\end{aligned}$$

Thus, $\mathcal{F}$ has a unique fixed point w in $C([0,\tau];L^2_\beta)$ and w solves W-problem in a weak sense in $(0,\tau]\times B$, if $C\tau<1$. We are able to continue this procedure to obtain the global well-posedness for the above constant C is independent of τ.

For the fixed point w, (4.4) and the boundedness of $\mathcal{A}$ imply that for $t\in[0,\tau']$

$$\begin{aligned}\max_{0\le t\le\tau'}||w(t,\cdot)||_{L^2_\beta}+||w||_{L^2(0,\tau';\overset{\circ}{H}{}^1_\beta)}+||\partial_t w||_{L^2(0,\tau';(\overset{\circ}{H}{}^1_\beta)^*)}\\ \le C||\mathcal{A}(w)||_{L^2(0,\tau';L^2_{2-\beta})}+C||w_0||_{L^2_\beta}\\ \le C\tau'\max_{0\le t\le\tau'}||w(t,\cdot)||_{L^2_\beta}+C||w_0||_{L^2_\beta}.\end{aligned}$$

We select a small $\tau'<T$ such that $C\tau'<1$. Then

$$\max_{0\le t\le\tau'}||w(t,\cdot)||_{L^2_\beta}+||w||_{L^2(0,\tau';\overset{\circ}{H}{}^1_\beta)}+||\partial_t w||_{L^2(0,\tau';(\overset{\circ}{H}{}^1_\beta)^*)}\le C||w_0||_{L^2_\beta}.$$

Thus,

$$||w(\tau',\cdot)||_{L^2_\beta}\le C||w_0||_{L^2_\beta}$$

and

$$\begin{aligned}\max_{\tau'\le t\le 2\tau'}||w(t,\cdot)||_{L^2_\beta}+||w||_{L^2(\tau',2\tau';\overset{\circ}{H}{}^1_\beta)}+||\partial_t w||_{L^2(\tau',2\tau';(\overset{\circ}{H}{}^1_\beta)^*)} &\le C||w(\tau',\cdot)||_{L^2_\beta}\\ &\le C^2||w_0||_{L^2_\beta}.\end{aligned}$$

Continuing, after finitely many steps we obtain an energy estimation similar to (4.4). The proof of Lemma 8 is thus complete.

5. Well-posedness for the FPF problem

In Section 3, we transformed the FPF problem to W-problem formally, but it is not difficult to show that they are equivalent. Indeed, one can verify that boundary condition (1.6) in the sense of (1.11) for the FPF problem is equivalent to the null boundary condition for W-problem.

For any test function $\varphi\in C^1_c$, the weak solution formulation for f can be transformed to the weak solution formulation for w, with $\varphi\rho^{b/2-1}$ as the test function. This is valid since $\varphi\rho^{b/2-1}\in C^1_c$ is dense in $\overset{\circ}{H}{}^1_\beta$. Such a justification can be reversed, hence the FPF problem and W-problem are equivalent.

Now we seek the function space in which the weak solution f to the FPF problem belongs. Recall that $\beta=-b/2+2$. For fixed $t\in[0,T]$, (3.7) implies

$$\int_B|f|^2\rho^{-b/2}dm = \int_B|w|^2\rho^\beta dm, \tag{5.1}$$

$$\int_B(|f|^2+|\nabla f|^2)\rho^{-b/2}dm \le C\int_B(|w|^2+|\nabla w|^2)\rho^\beta dm. \tag{5.2}$$

Also, for $\phi\in H^1_{-b/2}$ we have

$$|(\partial_t f,\phi)_{H^1_{-b/2}}|=|(\partial_t w,\rho^{-1}\phi)_{H^1_\beta}|\le C\|\partial_t w\|_{(H^1_\beta)^*}\|\phi\|_{H^1_{-b/2}}. \tag{5.3}$$

The estimate of the weak solution, (1.15) follows from Lemma 8 together with (5.1)-(5.3). This finishes the proof of (i) of Theorem 2.

6. Non-uniqueness

In this section we show that (1.6) is sharp in the sense that more solutions can be constructed if a weaker condition is imposed — this is to prove Proposition 4.

It suffices to construct more than one solution to the Fokker-Planck equation with $f_0(m) = 0$ and the assumption

$$||fd^{-1}|_{\partial B_r}||_{L^2(\partial B_r)} \neq 0 \text{ as } r \to \sqrt{b} \text{ for } t \in I. \tag{6.1}$$

Here I is a nonzero measurable set. The idea is to consider a class of functions $g(t,m) \in W^{2,\infty}((0,T) \times B)$ such that $g(0,m) = 0$ and $g(t,m)|_{\partial B} \neq 0$ for $t > 0$ (e.g. $g(t,m) = t|m|^2$) and show that for each g the following problem has a solution.

$$\partial_t f + \nabla \cdot (\kappa m f) = \frac{1}{2}\nabla \cdot \left(\frac{bm}{\rho} f\right) + \frac{1}{2}\Delta f, \qquad \text{in } (0,T] \times B, \tag{6.2}$$

$$f(0,m) = 0, \qquad m \in B, \tag{6.3}$$

$$f(t,m)\rho^{-1} = g(t,m), \qquad \text{in } (0,T] \times \partial B. \tag{6.4}$$

Note that $\beta = -b/2 + 2 < 1$, we can choose a parameter γ such that

$$\max\{\beta, -1\} < \gamma < 1. \tag{6.5}$$

To proceed, we define

$$w = \frac{f}{\rho} - g.$$

The resulting equation when multiplied by $\rho^{1-\gamma}$ leads to the following

$$\partial_t w\rho^\gamma - \tfrac{1}{2}\nabla \cdot (\nabla w \rho^\gamma) + (\beta - \gamma) m \cdot \nabla w \rho^{\gamma-1} + \kappa m \cdot \nabla w \rho^\gamma - \tilde{h}_0 = 0, \tag{6.6}$$

$$w(0,m) = 0, \; m \in B, \tag{6.7}$$

$$w(t,m) = 0, \; (t,m) \in [0,T] \times \partial B, \tag{6.8}$$

where

$$\tilde{h}_0(t,m) = cw\rho^{\gamma-1} - \partial_t g\rho^\gamma + \frac{1}{2}\nabla \cdot (\nabla g \rho^\gamma) - (\beta - \gamma) m \cdot \nabla g \rho^{\gamma-1} - \kappa m \cdot \nabla g \rho^\gamma + cg\rho^{\gamma-1}$$

with

$$c(t,m) = 2m \cdot \kappa(t) m + n(b/2 - 1).$$

Let

$$\begin{aligned} \mathcal{A}_0 : \quad L^2(0,\tau; L^2_\gamma) &\to L^2(0,\tau; L^2_{2-\gamma}) \\ w \quad &\mapsto \tilde{h}_0. \end{aligned}$$

This is well defined since $\gamma > -1$ from (6.5) and the assumption that $g \in W^{2,\infty}((0,T) \times B)$. From the same argument as that in Section 4, it follows that (6.6)-(6.8) has a unique solution w such that

$$w(t,m) \in L^2(0,T; \mathring{H}^1_\gamma), \quad \partial_t w(t,m) \in L^2(0,T; (\mathring{H}^1_\gamma)^*),$$

provided the corresponding U-problem

$$\partial_t u\rho^\gamma - \tfrac{1}{2}\nabla \cdot (\nabla u \rho^\gamma) + (\beta - \gamma) m \cdot \nabla u \rho^{\gamma-1} + \kappa m \cdot \nabla u \rho^\gamma - h_0 = 0, \tag{6.9}$$

$$u(0,m) = 0, \; m \in B, \tag{6.10}$$

$$u(t,m) = 0, \; (t,m) \in [0,T] \times \partial B, \tag{6.11}$$

has a solution for any $h_0 \in L^2(0,T;L^2_{2-\gamma})$. Note that $\gamma < 1$ is essential in order that the trace of w at the boundary is defined. Equation (6.9) is of the form of (4.1) but with an additional term $(\beta-\gamma)m\cdot\nabla u\rho^{\gamma-1}$. We thus define

$$\mathbf{L}_0[u,\varphi;t] \overset{\text{def.}}{=} \frac{1}{2}\int_B \nabla u\cdot\nabla\varphi\rho^\gamma dm + (\beta-\gamma)\int_B m\cdot\nabla u\varphi\rho^{\gamma-1}dm + \int_B \kappa m\cdot\nabla u\varphi\rho^\gamma dm.$$

We may obtain the existence and uniqueness for (6.9)-(6.11) from the same argument of the well-posedness for U-problem (4.1)-(4.3), if there is an energy estimate of $\mathbf{L}_0[u,u;t]$ which is similar to $\mathbf{L}[u,u;t]$ in Lemma 7. Indeed, for $u \in L^2(0,T;\overset{\circ}{H}{}^1_\gamma)$

$$\frac{1}{2}\int_B |\nabla u|^2\rho^\gamma dm = \mathbf{L}_0[u,u;t] - \frac{\beta-\gamma}{2}\left(\int_B m\cdot\nabla u^2\rho^{\gamma-1}dm\right) - \int_B \kappa m\cdot\nabla u u\rho^\gamma dm.$$

We now claim that

$$\int_B m\cdot\nabla u^2\rho^{\gamma-1}dm \le 0. \tag{6.12}$$

Given this together with $\gamma > \beta$ from (6.5) we have

$$\begin{aligned}\frac{1}{2}\int_B |\nabla u(\cdot,t)|^2\rho^\gamma dm &\le \mathbf{L}_0[u,u;t] - \int_B \kappa m\cdot\nabla u u\rho^\gamma dm\\ &\le \mathbf{L}_0[u,u;t] + \|\kappa\|_{L^\infty(0,T)}\sqrt{b}\left(\frac{1}{2\delta}\int_B|\nabla u|^2\rho^\gamma dm + \frac{\delta}{2}\int_B |u|^2\rho^\gamma dm\right)\end{aligned}$$

for any $\delta > 0$. By taking $\delta > \|\kappa\|_{L^\infty(0,T)}\sqrt{b}$, we obtain

$$C_1'\|u(t,\cdot)\|^2_{H^1_\gamma} \le \mathbf{L}_0[u,u;t] + C_2'\|u(t,\cdot)\|^2_{L^2_\gamma}$$

for appropriate constants C_1' and C_2'.

To verify the claim (6.12), we define the trace operator $\mathcal{T}_0$ such that

$$\begin{aligned}\mathcal{T}_0:\ \ H^1_\gamma(B) &\to L^2(\partial B)\\ u &\mapsto u\rho^{\frac{\gamma-1}{2}}|_{\partial B}.\end{aligned}$$

Integration by parts on (6.12) yields

$$\int_B m\cdot\nabla u^2\rho^{\gamma-1}dm = -\int_B u^2\left(n\rho^{\gamma-1} + 2(1-\gamma)|m|^2\rho^{\gamma-2}\right)dm + \sqrt{b}\int_{\partial B} u^2\rho^{\gamma-1}dS,$$

or

$$\begin{aligned}\sqrt{b}\int_{\partial B} u^2\rho^{\gamma-1}dS &= 2\int_B m\cdot\nabla u u\rho^{\gamma-1}dm + \int_B u^2\left(n\rho^{\gamma-1} + 2(1-\gamma)|m|^2\rho^{\gamma-2}\right)dm\\ &\le C\|u\|^2_{H^1_\gamma}.\end{aligned}$$

Thus $\mathcal{T}_0$ is well defined, and for $u \in \overset{\circ}{H}{}^1_\gamma$, $\mathcal{T}_0(u) = 0$. Finally we obtain

$$\int_B m\cdot\nabla u^2\rho^{\gamma-1}dm = -\int_B u^2\left(n\rho^{\gamma-1} + 2(1-\gamma)|m|^2\rho^{\gamma-2}\right)dm \le 0.$$

This shows that there is a unique weak solution u of (6.9)-(6.11), and thus $w \in L^2(0,T;\overset{\circ}{H}{}^1_\gamma)$ of (6.6)-(6.8).

Finally, $f = (w+g)\rho$ is a solution of (6.2)-(6.4) satisfying (6.1) for each g. Hence the uniqueness of (6.1)-(6.3) fails as stated in Proposition 4.

7. Conclusions

In this paper, we have identified a sharp Dirichlet-type boundary requirement to establish global existence of weak solutions to the microscopic FENE model which is a component of bead-spring type Navier-Stokes-Fokker-Planck models for dilute polymeric fluids. Such a boundary requirement states that the distribution near boundary approaches zero faster than the distance function. With this condition, we have been able to show the uniqueness of weak solutions in the weighted Sobolev space $H^1_{-b/2}(B)$, which when $b > 2$ is equivalent to the widely adopted weighted function space $\rho^{b/2}H^1_{b/2}(B)$ for Fokker-Planck equations with the FENE potential. Moreover, this condition ensures that the distribution remains a probability density. The sharpness of the boundary condition was shown through construction of infinitely many solutions when the boundary requirement fails. In other words, such a condition provides a threshold on the boundary requirement: subject to this condition or any stronger ones incorporated through a weighted function space, the Fokker-Planck dynamics will select the physically relevant solution, which is a probability density, see e.g. [1, 14, 2, 20, 24, 15], and converges to the equilibrium solution $Z^{-1}\rho^{b/2}$ [14]; any weaker boundary requirement will lead to more solutions. A detailed elaboration of boundary conditions for the coupled Navier-Stokes-Fokker-Planck model is presented in [22].

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