Hypocoercivity and Uniform Regularity for the Vlasov-Poisson-Fokker-Planck System with Uncertainty and Multiple Scales *

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Abstract

We study the Vlasov-Poisson-Fokker-Planck system with uncertainty and multiple scales. Here the uncertainy, modeled by random variables, enters the solution through initial data, while the multiple scales lead the system to its high-field or parabolic regimes. With the help of proper Lyapunov-type inequalities, under some mild conditions on the initial data, the regularity of the solution in the random space, as well as exponential decay of the solution to the global Maxwellian, are established under Sobolev norms, which are *uniform* in terms of the scaling parameters. These are the first hypocoercivity results for a nonlinear kinetic system with random input, which are important for the understanding of the sensitivity of the system under random perturbations, and for the establishment of spectral convergence of popular numerical methods for uncertainty quantification based on (spectrally accurate) polynomial chaos expansions.

Key words. Vlasov-Poisson-Fokker-Planck system, Uncertainty Quantification, random input, hypocoercivity

AMS subject classifications.

1 Introduction

In this paper we are interested in the Vlasov-Poisson-Fokker-Planck (VPFP) system with random inputs. The VPFP system describes the Brownian motion of a large system of particles in a surrounding bath. One of the applications is in electrostatic plasma, in which one considers the interactions between the electrons and a surrounding bath via the Coulomb force [4]. The uncertainty in a kinetic equation can arise from the initial and boundary data, the forcing term,

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collisional kernels, etc, due to modeling and measurement errors. In this paper we will mainly focus on the case in which the initial data contain random inputs, modeled by random variables with given probability density functions. The goal is to understand the regularity of the solution in the random space, as well as its long-time behavior. Such a study is important in order to understand the *sensitivity* of the system under random perturbations. It is also the basis to study the convergence of numerical schemes for such problems, for example, the popular methods for uncertainty quantification, such as polynomial chaos expansion based stochstic Galerkin or stochastic collocation methods [10, 13, 29, 28], which enjoy a spectral convergence, if the solution has the desired regularity in the random space.

While there have been many developments in the regularity of the solution to elliptic or parabolic equations with uncertainties [2, 5, 6], such study has been scarce for hyperbolic type equations [11, 26, 30, 3, 7] because of the poor regularity of the solution. The uncertainty quantification, while popular in many types of partial differential equations, has seldomly been studied for kinetic equations until very recently [31, 14, 21, 20]. Typically kinetic equatons possess multiple scales, leading to various different asymptotic regimes, demanding carefully designed numerical methods to handle different asymptotic behavior of the equations. For deterministic kinetic equations, one efficient multiscale paradigm is the *Asymptotic-Preserving* schemes, which mimic the asymptotic transitions from kinetic equations to their diffusion or hydrodynamic limits in the numerically discrete space [16, 17]. This concept was extended to random kinetic equations in [21], in the framework of *stochastic Asymptotic-Preserving* methods. Convergence study of these methods clearly require the understanding of the regularity of the solution. Moreover, the correct asymptotic behavior of the numerical methods in various asymptotic regimes also require the understanding of the long time behavior, as well as its rate of convergence toward the local or gloabl equilibrium. For linear transport equation with random isotropic scattering in diffusive regime, such regularity and asymptotic behavior were first studied in [18], in which the regularity of the solution was established, as well as its exponential decay toward the local equilibrium, all *uniformly* in the mean free path (or Knudsen number). Uniform regularity for the semiconductor Boltzmann equation, in which the scattering is anisotropic and random, was established in [19]. Called *hypocoercivity* by Villani [27], the property of uniform exponential decay toward the global equilibrium [8] was further explored in [22] for general linear kinetic equations with uncertainty. So far there has been no work on hypocoercivity for nonlinear kinetic equations with uncertainty with uniform (in small scaling parameters) estamite. The purpose of this paper is to conduct such a study for the nonlinear VPFP system with random initial input.

Depending on different scales, the VPFP system possesses two distinguished asymptotic limits, the high field limit and the parabolic limit. We will treat these different scalings in a unified framework. With the help of proper Lyapunov-type inequalities, we first develop two energy estimates for the microscopic (VPFP) and macroscopic (limiting) systems, which allows us to obtain the *uniform*–in terms of the scaling parameters–regularity in the random space of the perturbative solution of the nonlinear VPFP system near global Maxwellian. Under some mild conditions on the initial data, we found that the solution will exponentially decay to the global Maxwellian in a rate *independent* of the small scaling parameter. Our results also reveal that the initial random perturbation will die out exponentially in time, uniformly in the scaling parameter, thus the solution is insensitive to the initial random perturbation, in all asymptotic regimes.

For the deterministic VPFP system, the regularity and convergence toward the global Maxwellian or asymptotic limits were conducted in, for examples, [1, 9, 12, 23, 25, 15]. Our energy estimates rely on the hypocoercivity results of [9], and the energy estimates in [15] with suitable modification to effectively separate the microscopic and macroscopic scales in order to get better estimates in the small scaling parameter regimes. When the small scaling parameters are involved, which was not considered in [15], it is crucial to get rid of the bad dependence on these parameters in the initial condition and rate of convergence to the global equilibrium. Therefore we have not only extended the regularity results to the random space, but also improved the micro-macro energy estimates by separating the microscopic energy from the macroscopic energy suitably, so when the small scales are involved, we can get the optimal convergence rate towards the global equilibrium, and a milder initial condition at the same time. As a result, we get an exponential decay of the perturbative solution–independent of the small parameter– under some mild initial condition, which leads to a uniform regularity of the solution in random space for both high field and parabolic limits.

This paper is organized as follows. Section 2 gives an introduction of the VPFP system with uncertainty and its two different asymptotic regimes. The main results are stated in Section 3. Then in Sections 4 - 5 we prove the energy estimates from microscopic and macroscopic systems respectively. The uniform regularity of the perturbative solution is obtained in Section 6.

2 The VPFP System with Uncertainty and Asymptotic Scalings

2.1 The VPFP System with Uncertainty

In the dimensionless VPFP system with uncertainty, the time evolution of particle density distribution function $f(t, \mathbf{x}, \mathbf{v}, \mathbf{z})$ under the action of an electrical potential $\phi(t, \mathbf{x}, \mathbf{z})$ satisfies

$$\begin{cases} \partial_t f + \frac{1}{\delta}\mathbf{v}\cdot\bigtriangledown_{\mathbf{x}} f - \frac{1}{\epsilon}\bigtriangledown_{\mathbf{x}}\phi\cdot\bigtriangledown_{\mathbf{v}} f = \frac{1}{\delta\epsilon}\mathcal{F}f, \\ -\bigtriangleup_{\mathbf{x}}\phi = \rho - 1, \quad t>0, \quad \mathbf{x}\in\Omega\subset\mathbb{R}^N,\ \mathbf{v}\in\mathbb{R}^N,\ \mathbf{z}\in I_{\mathbf{z}}, \end{cases} \tag{2.1}$$

with periodic boundary condition in $\mathbf{x}$, and, initial data:

$$f(0, \mathbf{x}, \mathbf{v}, \mathbf{z}) = f^0(\mathbf{x}, \mathbf{v}, \mathbf{z}), \quad \mathbf{x}\in\Omega,\ \mathbf{v}\in\mathbb{R}^N,\ \mathbf{z}\in I_{\mathbf{z}}. \tag{2.2}$$

The distribution function $f(t, \mathbf{x}, \mathbf{v}, \mathbf{z})$ depends on time t, position $\mathbf{x}$, velocity $\mathbf{v}$ and random variable $\mathbf{z}\in I_{\mathbf{z}}\subseteq\mathbb{R}^d$. $\phi(t, \mathbf{x}, \mathbf{z})$ is a self-consistent electrical potential and $\rho(t, \mathbf{x}, \mathbf{z})$ is the density function defined as

$$\rho(t, \mathbf{x}, \mathbf{z}) = \int_{\mathbb{R}^N} f(t, \mathbf{x}, \mathbf{v}, \mathbf{z})d\mathbf{v}. \tag{2.3}$$

In the VPFP system, $\mathcal{F}$ is a collision operator, describing the Brownian motion of the particles, which reads,

$$\mathcal{F}f = \nabla\cdot\left(M\nabla\left(\frac{f}{M}\right)\right), \tag{2.4}$$

where M is the *global equilibrium* or *global Maxwellian*,

$$M = \frac{1}{(2\pi)^{\frac{N}{2}}} e^{-\frac{|\mathbf{v}|^2}{2}}. \tag{2.5}$$

In the dimensionless system, δ is the reciprocal of the scaled thermal velocity, ϵ represents the scaled thermal mean free path [25]. There are two different regimes for this system. One is the *high field regime*, where $\delta = 1$. As ϵ goes to zero, f goes to the local Maxwellian $M_{\text{local}} = \frac{1}{(2\pi)^{\frac{N}{2}}} e^{-\frac{|\mathbf{v}-\nabla_{\mathbf{x}}\phi|^2}{2}}$, and the VPFP system converges to a hyperbolic limit [1, 12, 23]:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \nabla_{\mathbf{x}} \phi) = 0, \\ -\triangle_{\mathbf{x}}\, \phi = \rho - 1, \end{cases} \tag{2.6}$$

Another regime is the *parabolic regime*, where $\delta = \epsilon$. When ϵ goes to zero, f goes to the global Maxwellian M, and the VPFP system converges to a parabolic limit [24]:

$$\begin{cases} \partial_t \rho - \nabla \cdot (\nabla_{\mathbf{x}} \rho - \rho \nabla_{\mathbf{x}} \phi) = 0, \\ -\triangle_{\mathbf{x}}\, \phi = \rho - 1. \end{cases} \tag{2.7}$$

In this paper, we are going to study both regimes together.

In the VPFP system with uncertainty, the random variable $\mathbf{z}$ is in a properly defined probability space $(\Sigma, \mathbb{A}, \mathbb{P})$, whose event space is Σ and is equipped with σ-algebra $\mathcal{A}$ and probability measure $\mathbb{P}$. Define $\pi(\mathbf{z}) : I_{\mathbf{z}} \longrightarrow \mathbb{R}^+$ as the probability density function of the random variable $\mathbf{z}(\omega)$, $\omega \in \Sigma$. So one has a corresponding L^2 space in the measure of,

$$d\mu = d\mu(\mathbf{x}, \mathbf{v}, \mathbf{z}) = \pi(\mathbf{z}) d\mathbf{x} d\mathbf{v} d\mathbf{z}. \tag{2.8}$$

With this measure, one has the corresponding Hilbert space with the following inner product and norms:

$$\langle f, g \rangle = \int_\Omega \int_{\mathbb{R}^N} \int_{I_{\mathbf{z}}} f g \, d\mu(\mathbf{x}, \mathbf{v}, \mathbf{z}), \quad \text{or,} \quad \langle \rho, j \rangle = \int_\Omega \int_{I_{\mathbf{z}}} \rho j \, d\mu(\mathbf{x}, \mathbf{z}), \quad \text{with norm } \|f\|^2 = \langle f, f \rangle. \tag{2.9}$$

For convenience of the readers, we list some elementary calculation on M which will be used in later calculationsi:

$$\partial_v M = -vM, \quad \partial_v(\sqrt{M}) = -\frac{v}{2}\sqrt{M}; \tag{2.10}$$

$$\int_{\mathbb{R}} v^a \sqrt{M}\, dv = \int v^a M\, dv = 0, \quad \text{for any odd } a; \tag{2.11}$$

$$\int_{\mathbb{R}} M\, dv = 1, \quad \int_{\mathbb{R}} v^2 M\, dv = 1, \quad \int_{\mathbb{R}} v^4 M dv = 3; \tag{2.12}$$

$$\int_{\mathbb{R}} |v|^3 M\, dv = \frac{4}{\sqrt{2\pi}} \leq 2, \quad \int_{\mathbb{R}} \left(\partial_v (v\sqrt{M})\right)^2 dv = \frac{3}{4}. \tag{2.13}$$

2.2 Notations

In this paper, we only focus on one space dimension. Without loss of generality, we assume $\epsilon < 1$, and $x \in \Omega = [0, l]$. For higher dimension of $\mathbf{x}, \mathbf{v}, \mathbf{z}$, it is easy to extend.

In order to get the convergence rate of the solution to the global equilibrium, we define,

$$h = \frac{f - M}{\sqrt{M}}, \quad \sigma = \int_{\mathbb{R}} h\sqrt{M}\, dv, \quad u = \int_{\mathbb{R}} h\, v\sqrt{M}\, dv, \tag{2.14}$$

where h is the fluctuations around the equilibrium, σ is the density fluctuation, u is the velocity fluctuation. Then the microscopic quantity h satisfies,

$$\begin{cases} \epsilon\delta \underbrace{\partial_t h}_{I} + \epsilon \underbrace{v\partial_x h}_{II} - \delta \underbrace{\partial_x\phi\partial_v h}_{III} + \delta \underbrace{\frac{v}{2}\partial_x\phi h}_{IV} + \delta \underbrace{v\sqrt{M}\partial_x\phi}_{V} = \underbrace{\mathcal{L}h}_{VI}, & (2.15) \\ \partial_x^2\phi = -\sigma, & (2.16) \end{cases}$$

where $\mathcal{L}$ is the so-called linearized Fokker-Planck operator,

$$\mathcal{L}h = \frac{1}{\sqrt{M}}\mathcal{F}\left(M + \sqrt{M}h\right) = \frac{1}{\sqrt{M}}\partial_v\left(M\partial_v\left(\frac{h}{\sqrt{M}}\right)\right). \tag{2.17}$$

We give each term a number, in order to make it clear where the term comes from originally when doing the energy estimates later.

We further introduce projections onto $\sqrt{M}$ and $v\sqrt{M}$,

$$\Pi_1 h = \sigma\sqrt{M}, \quad \Pi_2 h = uv\sqrt{M}, \quad \Pi h = \Pi_1 h + \Pi_2 h. \tag{2.18}$$

These projections have the following properties:

- $\partial_x\partial_z\Pi = \Pi\partial_x\partial_z$
- Due to the mutual orthogonality of $\Pi_1 h$, $\Pi_2 h$, $(1-\Pi)h$ in L_v^2 space, let $\partial^{\mathbf{k}} = \partial_z^{k_1}\partial_x^{k_2}$,

$$\begin{aligned} \left\|\partial^{\mathbf{k}} h\right\|^2 &= \left\|\Pi_1\partial^{\mathbf{k}} h\right\|^2 + \left\|\Pi_2\partial^{\mathbf{k}} h\right\|^2 + \left\|(1-\Pi)\partial^{\mathbf{k}} h\right\|^2 \\ &= \left\|\partial^{\mathbf{k}}\sigma\right\|^2 + \left\|\partial^{\mathbf{k}} u\right\|^2 + \left\|(1-\Pi)\partial^{\mathbf{k}} h\right\|^2, \end{aligned} \tag{2.19}$$

  which also implies,

$$\left\|\partial^{\mathbf{k}}\sigma\right\|, \left\|\partial^{\mathbf{k}} u\right\|, \left\|(1-\Pi)\partial^{\mathbf{k}} h\right\| \le \left\|\partial^{\mathbf{k}} h\right\|. \tag{2.20}$$

Multiplying $\sqrt{M}$ and $v\sqrt{M}$ to (2.15), and integrating the equation over v respectively, then one has the equations for the macroscopic quantities σ and u,

$$\begin{cases} \delta\partial_t\sigma + \partial_x u = 0, & (2.21) \\ \epsilon\delta \underbrace{\partial_t u}_{I} + \epsilon \underbrace{\partial_x\sigma}_{II.1} + \epsilon \underbrace{\int v^2\sqrt{M}(1-\Pi)\partial_x h dv}_{II.2} + \delta \underbrace{\partial_x\phi\sigma}_{III} + \underbrace{u}_{VI} + \delta \underbrace{\partial_x\phi}_{V} = 0. & (2.22) \end{cases}$$

We call (2.15)-(2.16) the *microscopic system*, and (2.21)-(2.22) the *macroscopic system*. Note (2.21)-(2.22) are not a closed system since it contains the microscopic quantities h.

We also define the following norms and energies,

- Norms:
  - $\|h\|_{L^2(v)}^2 = \int_{\mathbb{R}} h^2\, dv$,

- $\|f\|_{H^m}^2 = \sum_{l=0}^m \left\|\partial_z^l f\right\|^2$, $\quad \|f\|_{H^1(x,z)}^2 = \|f\|^2 + \|\partial_x f\|^2 + \|\partial_z f\|^2$,
- $\|h\|_v^2 = \int_{(0,l)\times\mathbb{R}\times I_z} (\partial_v h)^2 + (1+|v|^2)h^2 \, d\mu(x,v,z)$, $\quad \|h\|_{H_v^m}^2 = \sum_{l=0}^m \left\|\partial_z^l h\right\|_v^2$;

- Energy terms:
  - $E_h^m = \|h\|_{H^m}^2 + \|\partial_x h\|_{H^{m-1}}^2$, $\quad E_\phi^m = \|\partial_x \phi\|_{H^m}^2 + \left\|\partial_x^2 \phi\right\|_{H^{m-1}}^2$;
- Dissipation terms:
  - $D_h^m = \|(1-\Pi)h\|_{H^m}^2 + \|(1-\Pi)\partial_x h\|_{H^{m-1}}^2$, $\quad D_\phi^m = E_\phi^m$,
  - $D_u^m = \|u\|_{H^m}^2 + \|\partial_x u\|_{H^{m-1}}^2$, $\quad D_\sigma^m = \|\sigma\|_{H^m}^2 + \|\partial_x \sigma\|_{H^{m-1}}^2$.

3 Main Results

To get the regularity of the solution in the Hilbert space, one usually uses energy estimates. In order to balance the nonlinear term $\partial_x\phi\partial_v f$, and get a regularity independent of the small parameter ϵ (or depending on ϵ in a good way), one needs the hypocoercivity property from the collision operator. The hypocoercivity property one uses most commonly is

$$-\langle \mathcal{L}h, h\rangle \geq C\,\|(1-\Pi_1)h\|^2\,, \tag{3.1}$$

see [8, 27]. However, this is not enough for the non-linear case. We need stronger hypocoercivity as listed in the following Proposition, see [9] for details.

Proposition 3.1. *For $\mathcal{L}$ defined in (2.4),*

(a) $-\langle \mathcal{L}h, h\rangle = -\langle L(1-\Pi)h, (1-\Pi)h\rangle + \|u\|^2$;

(b) $-\langle \mathcal{L}(1-\Pi)h, (1-\Pi)h\rangle = \|\partial_v(1-\Pi)h\|^2 + \frac{1}{4}\|v(1-\Pi)h\|^2 - \frac{1}{2}\|(1-\Pi)h\|^2$;

(c) $-\langle \mathcal{L}(1-\Pi)h, (1-\Pi)h\rangle \geq \|(1-\Pi)h\|^2$;

(d) There exists a constant $\lambda_0 > 0$, such that the following hypocoercivity holds,

$$-\langle \mathcal{L}h, h\rangle \geq \lambda_0\,\|(1-\Pi)h\|_v^2 + \|u\|^2\,, \tag{3.2}$$

and the largest $\lambda_0 = \frac{1}{7}$ in one dimension.

Proof. Here we only prove (d), see [9] for $(a), (b), (c)$. Since

$$\begin{aligned} &-\langle \mathcal{L}(1-\Pi)h, (1-\Pi)h\rangle \\ \geq& a\,\|\partial_v(1-\Pi)h\|^2 + \frac{a}{4}\|v(1-\Pi)h\|^2 - \frac{a}{2}\|(1-\Pi)h\|^2 + (1-a)\,\|(1-\Pi)h\|^2 \\ \geq& \min_{0<a<1}\{a, \frac{a}{4}, (1-\frac{3}{2}a)\}\,\|(1-\Pi)h\|_v^2\,, \end{aligned} \tag{3.3}$$

then the largest λ_0 one can get is when $a = \frac{4}{7}$, $\lambda_0 = \frac{1}{7}$. Therefore,

$$-\langle \mathcal{L}h, h\rangle \geq \lambda_0\,\|(1-\Pi)h\|_v^2 + \|u\|^2\,. \tag{3.4}$$

∎

Based on the hypocoercivity (3.2), we have the following two estimates for the microscopic and macroscopic systems respectively.

Lemma 3.2. *The solution to system (2.15) - (2.16) satisfies the following estimates,*

$$\begin{aligned}&\frac{\delta}{2}\partial_t\left[\epsilon E_h^m+\delta E_\phi^m\right]+\lambda_0 D_h^m+D_u^m\\ \leq &AC_1\delta\left(\sqrt{E_h^m}+2\sqrt{E_\phi^m}\right)(5D_u^m+4D_h^m)+AC_1\delta\sqrt{E_h^m}D_\phi^m+AC_1\delta\sqrt{E_\phi^m}D_\sigma^m,\end{aligned}\tag{3.5}$$

and

$$\begin{aligned}&\delta\partial_t\left[\epsilon\sum_{l=0}^{m-1}\langle\partial_z^l u,\partial_z^l\partial_x\phi\rangle+\epsilon\sum_{l=0}^{m}\langle\partial_z^l\partial_x u,\partial_z^l\partial_x^2\phi\rangle+\frac{1}{2}E_\phi^m\right]+\frac{\epsilon}{2}D_\sigma^m+\delta D_\phi^m\\ \leq&\epsilon D_u^m+\frac{\epsilon}{2}D_h^m+2AC_1\delta\sqrt{E_h^m}\,D_\phi^m+AC_1\delta\sqrt{E_\phi^m}\,D_\sigma^m,\end{aligned}\tag{3.6}$$

where

$$A=2\binom{m}{[m/2]}\tag{3.7}$$

is a constant only depending on m, $[m/2]$ is the smallest integer larger or equal to $\frac{m}{2}$, and C_1 is the Sobolev constant in one dimension defined in (A.40).

If one combines the above two inequalities, the "bad terms" on the right hand side (RHS) can be controlled by the dissipation terms on the left hand side (LHS) if the coefficients are carefully balanced. Hence, one can come to the conclusion that the solution exponentially decays to the global equilibrium.

Remark 3.3. *The main difference between the energy estimates in Lemma 3.2 and the one obtained in [15] is that for both micro and macro systems, we separate the microscopic energy E_h^m from the macroscopic energy E_ϕ^m for D_ϕ^m and D_σ^m, which gives us more flexibility to bound the energies, especially when small parameters are involved.*

Theorem 3.4. *For the high field regime ($\delta=1$), if*

$$E_h^m(0)+\frac{1}{\epsilon^2}E_\phi^m(0)\leq\frac{2\lambda_0^3}{(80AC_1)^2},\tag{3.8}$$

then,

$$E_h^m(t)+\frac{1}{\epsilon^2}E_\phi^m(t)\leq\frac{3}{\lambda_0}e^{-\frac{\lambda_0}{3}t}\left(E_h^m(0)+\frac{1}{\epsilon^2}E_\phi^m(0)\right).\tag{3.9}$$

For the parabolic regime ($\delta=\epsilon$), if

$$E_h^m(0)+\frac{1}{\epsilon}E_\phi^m(0)\leq\frac{2\lambda_0^3}{(80AC_1)^2},\tag{3.10}$$

then,

$$E_h^m(t)+\frac{1}{\epsilon}E_\phi^m(t)\leq\frac{3}{\lambda_0}e^{-\frac{\lambda_0}{2}t}\left(E_h^m(0)+\frac{1}{\epsilon}E_\phi^m(0)\right).\tag{3.11}$$

Here A and C_1 are the same as in Lemma 3.2.

Remark 3.5. *Basically, Theorem 3.4 implies the following,*

(a) As long as initially the electric field $\partial_x \phi$ is small enough ($O(\epsilon)$ or $O(\sqrt{\epsilon})$), and the initial data f is suitably bounded by (3.8) or (3.10), then the solution will converge to the global equilibrium exponentially, uniformly in ϵ.

(b) The initial data for the distribution only requires $f - M = O(1)$, which is independent of ϵ for both cases. If one directly applies the conclusion of [15], then for the high field regime, $E_h^m(0)$ and $E_\phi^m(0)$ need to be $O(\epsilon)$ and $O(\epsilon^3)$ initially, see Remark 6.1 for details. Our result allows more general initial data for f while keeping the optimal convergence rate at the same time, which is because of the new energy estimates we obtained in Lemma 3.2.

Besides all the conclusions above, one can obtain the regularity for the solution to VPFP system immediately from Theorem 3.4.

Theorem 3.6. *Under the same condition given in Theorem 3.4, one has*

$$\|f(t)\|_{H^m}^2 \leq \frac{3}{(20AC_1)^2} + 2l^2. \tag{3.12}$$

Proof.

$$\|f(t)\|_{H^m}^2 \leq \left\|\frac{f}{\sqrt{M}}\right\|_{H^m}^2 = \left\|\frac{f-M}{\sqrt{M}} + \sqrt{M}\right\|_{H^m}^2 \leq 2E_h^m + 2\left\|\sqrt{M}\right\|^2 \leq 2\frac{6}{\lambda_0}\frac{\lambda_0}{(40AC_1)^2} + 2l^2. \tag{3.13}$$

■

Remark 3.7. *From Theorem 3.6, one knows the regularity of the initial data in the random space is preserved in time. Furthermore, the bound is independent of the small parameter ϵ.*

3.1 Some Inequalities

In this section, we give some equalities and inequalities that will be frequently used later.

Lemma 3.8. *Let $\partial^{\mathbf{k}} = \partial_z^{k_1}\partial_x^{k_2}$, and similar for $\partial^{\mathbf{i}}, \partial^{\mathbf{l}}$,*

(a) $\left\langle \partial^{\mathbf{k}}\partial_x\phi, v\sqrt{M}\partial^{\mathbf{k}}h \right\rangle = \frac{\delta}{2}\partial_t \left\|\partial^{\mathbf{k}}\partial_x\phi\right\|^2,$

(b) $$\langle \partial^{\mathbf{k}}\partial_x\phi\, \partial_v(\partial^{\mathbf{i}}h), \partial^{\mathbf{l}}h\rangle - \frac{1}{2}\langle v\partial^{\mathbf{k}}\partial_x\phi\, \partial^{\mathbf{i}}h, \partial^{\mathbf{l}}h\rangle$$
$$\leq C_1 \left\|\partial^{\mathbf{k}}\partial_x\phi\right\|_{H^1(x,z)} \left(\left\|\partial^{\mathbf{i}}\sigma\right\|^2 + 2\left\|\partial^{\mathbf{i}}u\right\|^2 + 2\left\|(1-\Pi)\partial^{\mathbf{i}}h\right\|_v^2 + 3\left\|\partial^{\mathbf{l}}u\right\|^2 + 2\left\|(1-P)\partial^{\mathbf{l}}h\right\|_v^2\right),$$

(c) $$\langle \partial^{\mathbf{k}}\partial_x\phi\, \partial_v(\partial^{\mathbf{i}}h), \partial^{\mathbf{l}}h\rangle - \frac{1}{2}\langle v\partial^{\mathbf{k}}\partial_x\phi\partial^{\mathbf{i}}h, \partial^{\mathbf{l}}h\rangle$$
$$\leq C_1 \left\|\partial^{\mathbf{i}}h\right\|_{H^1(x,z)} \left(3\left\|\partial^{\mathbf{l}}u\right\|^2 + 2\left\|(1-\Pi)\partial^{\mathbf{l}}h\right\|_v^2 + \left\|\partial^{\mathbf{k}}\partial_x\phi\right\|^2\right),$$

(d) $\left\|\partial^{\mathbf{k}}\partial_x\partial_t\phi\right\|^2 \leq \frac{1}{\delta^2}\left\|\partial^{\mathbf{k}}u\right\|^2.$

Proof. See Appendix. ■

Remark 3.9. *Notice that in the inequalities* (b) *and* (c)*, the dissipations of* u *and* $(1-\Pi)h$ *are related to both energies* h *and* $\partial_x\phi$*. However, the dissipation of* σ *is only related to the energy of* $\partial_x\phi$ *through* (b)*, while the dissipation of* $\partial_x\phi$ *is only related to the energy of* h *through* (c)*. This is why we can get the separation of the micro and macro energies in Lemma 3.2 for* D^m_σ *and* D^m_ϕ*.*

4 Energy Estimates on the Microscopic Equations

Now we prove the first part of Lemma 3.2, (3.5).

4.1 Energy estimates for $\left\|\partial_z^l h\right\|^2$

Taking ∂_z^l to (2.15), and multiplying by $\partial_z^l h$, then integrating it over $\mu(x,v,z)$, one has,

$$\begin{aligned}
&\frac{\epsilon\delta}{2}\partial_t\left\|\partial_z^l h\right\|^2+\delta\underbrace{\langle\partial_z^l\partial_x\phi, v\sqrt{M}\partial_z^l h\rangle}_{V}\underbrace{-\langle\mathcal{L}\partial_z^l h,\partial_z^l h\rangle}_{VI}\\
=&\delta\sum_{i=0}^{l}\tbinom{l}{i}\left(\underbrace{\langle\partial_z^{l-i}\partial_x\phi\partial_v\partial_z^i h,\partial_z^l h\rangle}_{III}-\underbrace{\frac{1}{2}\langle v\partial_z^{l-i}\partial_x\phi\,\partial_z^i h,\partial_z^l h\rangle}_{IV}\right).
\end{aligned}\tag{4.1}$$

V and VI are "good terms", since by Lemma 3.8 (a) and Proposition 3.1 (d),

$$V=\frac{\delta}{2}\partial_t\left\|\partial_z^l\partial_x\phi\right\|^2,\quad VI\geq\lambda_0\left\|(1-\Pi)\partial_z^l h\right\|_v^2+\left\|\partial_z^l u\right\|^2.\tag{4.2}$$

However, III and IV are "bad terms" here, and one wants to control it by the dissapations.
For $i<l$, by Lemma 3.8 (c),

$$III-IV\leq C_1\left\|\partial_z^i h\right\|_{H^1(x,z)}\left(3\left\|\partial_z^l u\right\|^2+2\left\|(1-\Pi)\partial_z^l h\right\|_v^2+\left\|\partial_z^{l-i}\partial_x\phi\right\|^2\right).\tag{4.3}$$

For $i=l$, by Lemma 3.8 (b),

$$III-IV\leq C_1\left\|\partial_x\phi\right\|_{H^1(x,z)}\left\|\partial_z^l\sigma\right\|^2+C_1\left\|\partial_x\phi\right\|_{H^1(x,z)}\left(5\left\|\partial_z^l u\right\|^2+4\left\|(1-P)\partial_z^l h\right\|_v^2\right).\tag{4.4}$$

Here if one treats the case of $i=l$ the same as the case of $i<l$, then the largest $i=m$ leads to $\|\partial_z^m h\|_{H^1(x,z)}$, which cannot be controlled by $\partial_t E_h^m$, so we treat $i=l$ differently from $i<l$. Therefore one has the energy estimate,

$$\begin{aligned}
&\frac{\delta}{2}\partial_t\left(\epsilon\left\|\partial_z^l h\right\|^2+\delta\left\|\partial_z^l\partial_x\phi\right\|^2\right)+\lambda_0\left\|(1-\Pi)\partial_z^l h\right\|_v^2+\left\|\partial_z^l u\right\|^2\\
\leq&C_1\delta\sum_{l\neq 1,i=0}^{l-1}\tbinom{l}{i}\left\|\partial_z^i h\right\|_{H^1(x,z)}(3\left\|\partial_z^l u\right\|^2+2\left\|(1-\Pi)\partial_z^l h\right\|_v^2+\left\|\partial_z^{l-i}\partial_x\phi\right\|^2)\\
&+C_1\delta\sqrt{E_\phi^1}\left\|\partial_z^l\sigma\right\|^2+C_1\delta\sqrt{E_\phi^1}(5\left\|\partial_z^l u\right\|^2+4\left\|(1-\Pi)\partial_z^l h\right\|_v^2).
\end{aligned}\tag{4.5}$$

Summing l from 0 to m, one gets,

$$
\begin{aligned}
&\frac{\delta}{2}\partial_t\left[\epsilon\|h\|_{H^m}^2+\delta\|\partial_x\phi\|_{H^m}^2\right]+\lambda_0\|(1-\Pi)h\|_{H^m,v}^2+\|u\|_{H^m}^2\\
\leq& C_1\delta\sum_{l=1}^m\sum_{i=0}^{l-1}\binom{l}{i}\left\|\partial_z^i h\right\|_{H^1(x,z)}\left(3\left\|\partial_z^l u\right\|^2+2\left\|(1-\Pi)\partial_z^l h\right\|_v^2\right)+C_1\delta\sum_{l=1}^m\sum_{i=1}^m\binom{l}{i}\left\|\partial_z^{l-i}h\right\|_{H^1(x,z)}\left\|\partial_z^i\partial_x\phi\right\|^2\\
&+C_1\delta\sqrt{E_\phi^1}\,\|\sigma\|_{H^m}^2+C_1\delta\sqrt{E_\phi^1}\left(5\|u\|_{H^m}^2+4\|(1-\Pi)h)\|_{H_v^m}^2\right)\\
=&C_1\delta\sum_{l=1}^m\left(\sum_{i=0}^{l-1}\binom{l}{i}\left\|\partial_z^i h\right\|_{H^1(x,z)}\right)\left(3\left\|\partial_z^l u\right\|^2+2\left\|(1-\Pi)\partial_z^l h\right\|_v^2\right)\\
&+C_1\delta\sum_{i=1}^m\left(\sum_{l=i}^m\binom{l}{i}\left\|\partial_z^{l-i}h\right\|_{H^1(x,z)}\right)\left\|\partial_z^i\partial_x\phi\right\|^2\\
&+C_1\delta\sqrt{E_\phi^1}\,\|\sigma\|_{H^m}^2+C_1\delta\sqrt{E_\phi^1}\left(5\|u\|_{H^m}^2+4\|(1-\Pi)h)\|_{H_v^m}^2\right)\\
\leq& AC_1\delta\sqrt{E_h^m}\left(3\|u\|_{H^m}^2+2\left\|(1-\Pi)\partial_z^l h\right\|_{H_v^m}^2\right)+AC_1\delta\sqrt{E_h^m}\,\|\partial_x\phi\|_{H^m}^2+C_1\delta\sqrt{E_\phi^1}\,\|\sigma\|_{H^m}^2\\
&+C_1\delta\sqrt{E_\phi^1}\left(5\|u\|_{H^m}^2+4\left\|(1-\Pi)\partial_z^l h\right\|_{H_v^m}^2\right)\\
\leq& C_1\delta\left(A\sqrt{E_h^m}+\sqrt{E_\phi^1}\right)\left(5\|u\|_{H^m}^2+4\left\|(1-\Pi)\partial_z^l h\right\|_{H_v^m}^2\right)+AC_1\delta\sqrt{E_h^m}\,\|\partial_x\phi\|_{H^m}^2+C_1\delta\sqrt{E_\phi^1}\,\|\sigma\|_{H^m}^2\,,
\end{aligned}
\tag{4.6}
$$

where A is given by (3.7).

Remark 4.1. *Before we move on to other estimates, let us first summarize what else we need. The goal of the energy estimates is to get an inequality like*

$$
\partial_t E+D\leq\sqrt{E}D,\tag{4.7}
$$

so one can use the continuity argument to get the desired estimates. Therefore, one still needs $\partial_t\|\partial_x h\|_{H^{m-1}}^2$, $\|\sigma\|_{H^m}$, $\|\partial_x\phi\|_{H^m}^2$ *on the LHS.*

4.2 Energy estimates for $\left\|\partial_z^l\partial_x h\right\|^2$

Taking $\partial_z^l\partial_x$ to (2.15), and multiplying by $\partial_z^l\partial_x h$, then integrating it over $\mu(x,v,z)$,

$$
\begin{aligned}
&\frac{\epsilon\delta}{2}\partial_t\left\|\partial_z^l\partial_x h\right\|^2+\delta\underbrace{\left\langle\partial_z^l\partial_x^2\phi,v\sqrt{M}\partial_z^l\partial_x h\right\rangle}_{V}\underbrace{-\left\langle\mathcal{L}\partial_z^l\partial_x h,\partial_z^l\partial_x h\right\rangle}_{VI}\\
=&\delta\sum_{i=0}^l\binom{l}{i}\left\langle\underbrace{\partial_z^{l-i}\partial_x^2\phi\partial_v\partial_z^i h}_{III.1}+\underbrace{\partial_z^{l-i}\partial_x\phi\partial_v\partial_x\partial_z^i h}_{III.2}-\underbrace{\frac{v}{2}\partial_z^{l-i}\partial_x^2\phi\partial_z^i h}_{IV.1}-\underbrace{\frac{v}{2}\partial_z^{l-i}\partial_x\phi\partial_z^i\partial_x h}_{IV.2},\partial_z^l\partial_x h\right\rangle.
\end{aligned}
$$

Similar to (4.2), for V and VI, one has,

$$
V=\frac{\delta}{2}\partial_t\left\|\partial_z^l\partial_x^2\phi\right\|^2,\quad VI\geq\lambda_0\left\|(1-\Pi)\partial_z^l\partial_x h\right\|^2+\left\|\partial_z^l\partial_x u\right\|^2.\tag{4.8}
$$

For the bad terms on the RHS, by Lemma 3.8 (c),

$$
III.1-IV.1\leq C_1\left\|\partial_z^i h\right\|_{H^1(x,z)}\left(3\left\|\partial_z^l\partial_x u\right\|^2+2\left\|(1-\Pi)\partial_z^l\partial_x h\right\|^2+\left\|\partial_z^{l-i}\partial_x^2\phi\right\|^2\right).\tag{4.9}
$$

Remark 4.2. *If one treats $III.2 - IV.2$ the same as $III.1 - IV.1$, then $\left\|\partial_z^i \partial_x h\right\|_{H^1(x,z)}$ cannot be controlled. So one needs to treat it differently.*

By Lemma 3.8 (b),

$$\begin{aligned} III.2 - IV.2 \leq & C_1 \left\|\partial_z^{l-i} \partial_x \phi\right\|_{H^1(x,z)} \left(\left\|\partial_z^i \partial_x \sigma\right\|^2 + 2\left\|\partial_z^i \partial_x u\right\|^2 + 2\left\|(1-\Pi)\partial_z^i \partial_x h\right\|_v^2 \right. \\ & \left. + 3\left\|\partial_z^l \partial_x u\right\|^2 + 2\left\|(1-\Pi)\partial_z^l \partial_x h\right\|_v^2 \right). \end{aligned} \tag{4.10}$$

Combining all the terms gives,

$$\begin{aligned} & \frac{\delta}{2}\partial_t \left[\epsilon \left\|\partial_z^l \partial_x h\right\|^2 + \delta \left\|\partial_z^l \partial_x^2 \phi\right\|^2 \right] + \lambda_0 \left\|(1-\Pi)\partial_z^l \partial_x h\right\|_v^2 + \left\|\partial_z^l \partial_x u\right\|^2 \\ \leq & C_1 \delta \left(\sum_{i=0}^{l} \tbinom{l}{i} \left(\left\|\partial_z^i h\right\|_{H^1(x,z)} + \left\|\partial_z^{l-i} \partial_x \phi\right\|_{H^1(x,z)} \right) \right) (3\left\|\partial_z^l \partial_x u\right\|^2 + 2\left\|(1-\Pi)\partial_z^l \partial_x h\right\|_v^2) \\ & + C_1 \delta \sum_{i=0}^{l} \tbinom{l}{i} \left\|\partial_z^i h\right\|_{H^1(x,z)} \left\|\partial_z^{l-i} \partial_x^2 \phi\right\|^2 + C_1 \delta \sum_{i=0}^{l} \tbinom{l}{i} \left\|\partial_z^{l-i} \partial_x \phi\right\|_{H^1(x,z)} \left\|\partial_z^i \partial_x \sigma\right\|^2 \\ & + C_1 \delta \sum_{i=0}^{l} \tbinom{l}{i} \left\|\partial_z^{l-i} \partial_x \phi\right\|_{H^1(x,z)} (2\left\|\partial_z^i \partial_x u\right\|^2 + 2\left\|(1-\Pi)\partial_z^i \partial_x h\right\|_v^2). \end{aligned} \tag{4.11}$$

Summing l from 0 to $m-1$ gives,

$$\begin{aligned} & \frac{\delta}{2}\partial_t \left[\epsilon \left\|\partial_x h\right\|_{H^{m-1}}^2 + \delta \left\|\partial_x^2 \phi\right\|_{H^{m-1}}^2 \right] + \lambda_0 \left\|(1-\Pi)\partial_x h\right\|_{H_v^{m-1}}^2 + \left\|\partial_x u\right\|_{H^{m-1}}^2 \\ \leq & C_1 \delta \sum_{l=0}^{m-1} \left(\sum_{i=0}^{l} \tbinom{l}{i} \left(\left\|\partial_z^i h\right\|_{H^1(x,z)} + \left\|\partial_z^{l-i} \partial_x \phi\right\|_{H^1(x,z)} \right) + \sum_{i=l}^{m-1} \tbinom{i}{l} \left\|\partial_z^{i-l} \partial_x \phi\right\|_{H^1(x,z)} \right) \\ & \cdot (3\left\|\partial_z^l \partial_x u\right\|^2 + 2\left\|(1-\Pi)\partial_z^l \partial_x h\right\|_v^2) + C_1 \delta \sum_{i=0}^{m-1} \left(\sum_{l=i}^{m-1} \tbinom{l}{i} \left\|\partial_z^{l-i} h\right\|_{H^1(x,z)} \right) \left\|\partial_z^i \partial_x^2 \phi\right\|^2 \\ & + C_1 \delta \sum_{i=0}^{m-1} \left(\sum_{l=i}^{m-1} \tbinom{l}{i} \left\|\partial_z^{l-i} \partial_x \phi\right\|_{H^1(x,z)} \right) \left\|\partial_z^i \partial_x \sigma\right\|^2 \\ \leq & A C_1 \delta \left(\sqrt{E_h^m} + 2\sqrt{E_\phi^m} \right) \left(3\left\|\partial_x u\right\|_{H^{m-1}}^2 + 2\left\|(1-\Pi)\partial_x h\right\|_{H_v^{m-1}}^2 \right) \\ & + A C_1 \delta \sqrt{E_h^m} \left\|\partial_x^2 \phi\right\|_{H^{m-1}}^2 + A C_1 \delta \sqrt{E_\phi^m} \left\|\partial_x \sigma\right\|_{H^{m-1}}^2, \end{aligned} \tag{4.12}$$

where A is defined as (3.7). Now combining (4.6) and (4.12) completes the energy estimates for the microscopic system,

$$\begin{aligned} & \frac{\delta}{2}\partial_t \left[\epsilon \underbrace{E_h^m}_{I} + \delta \underbrace{E_\phi^m}_{V} \right] + \underbrace{\lambda_0 D_h^m + D_u^m}_{VI} \\ \leq & \underbrace{A C_1 \delta \left(\sqrt{E_h^m} + 2\sqrt{E_\phi^m} \right) (5 D_u^m + 4 D_h^m) + A C_1 \delta \sqrt{E_h^m} D_\phi^m + A C_1 \delta \sqrt{E_\phi^m} D_\sigma^m}_{III+IV}. \end{aligned} \tag{4.13}$$

Up to now, one still needs the dissipations D_σ^m and D_ϕ^m on LHS to balance the bad terms on RHS. So next we turn to the macroscopic system.

5 Energy Estimates on the Macroscopic System

We now prove (3.6) in Lemma 3.2.

5.1 Dissipation terms $\left\|\partial_z^l \sigma\right\|^2$ and $\left\|\partial_z^l \partial_x \phi\right\|^2$

Taking ∂_z^l to (2.22) and multiplying by $\partial_z^l \partial_x \phi$, then integrating it over $\mu(x,z)$, one has,

$$
\begin{aligned}
&\epsilon\delta\underbrace{\langle \partial_t \partial_z^l u, \partial_z^l \partial_x \phi\rangle}_{I} + \epsilon\underbrace{\langle \partial_z^l \partial_x \sigma, \partial_z^l \partial_x \phi\rangle}_{II.1} + \underbrace{\langle \partial_z^l u, \partial_z^l \partial_x \phi\rangle}_{VI} + \delta\underbrace{\left\|\partial_z^l \partial_x \phi\right\|^2}_{V} \\
&= -\epsilon\underbrace{\langle (1-\Pi)\partial_z^l \partial_x h, v^2\sqrt{M}\partial_z^l \partial_x \phi\rangle}_{II.2} - \delta\sum_{i=0}^{l}\binom{l}{i}\underbrace{\langle \partial_z^{l-i}\partial_x \phi \partial_z^i \sigma, \partial_z^l \partial_x \phi\rangle}_{III}.
\end{aligned}
\tag{5.1}
$$

First one has,

$$
I = \partial_t \langle \partial_z^l u, \partial_z^l \partial_x \phi\rangle - \langle \partial_z^l u, \partial_t \partial_z^l \partial_x \phi\rangle,
\tag{5.2}
$$

then by Lemma 3.8 (d),

$$
\langle \partial_z^l u, \partial_z^l \partial_x \partial_t \phi\rangle = \delta\langle \partial_z^l \partial_t \sigma, \partial_z^l \partial_t \phi\rangle = \delta\left\|\partial_z^l \partial_x \partial_t \phi\right\|^2 \le \frac{1}{\delta}\left\|\partial_z^l u\right\|^2.
\tag{5.3}
$$

$II.1$ and VI are "good terms" here, since

$$
II.1 = \langle \partial_z^l \sigma, -\partial_z^l \partial_x^2 \phi\rangle = \left\|\partial_z^l \sigma\right\|^2,
\tag{5.4}
$$

$$
VI = -\langle \partial_z^l \partial_x u, \partial_z^l \phi\rangle = \delta\langle \partial_z^l \partial_t \sigma, \partial_z^l \phi\rangle = \delta\langle \partial_z^l \partial_x \partial_t \phi, \partial_z^l \partial_x \phi\rangle = \frac{\delta}{2}\partial_t\left\|\partial_z^l \partial_x \phi\right\|^2,
\tag{5.5}
$$

while $II.2$ and III are "bad terms",

$$
\begin{aligned}
-II.2 =& \langle (1-\Pi)\partial_z^l h, v^2\sqrt{M}\partial_z^l \partial_x^2 \phi\rangle \le \frac{1}{2}\left\|v\sqrt{M}\partial_z^l \sigma\right\|^2 + \frac{1}{2}\left\|v(1-\Pi)\partial_z^l h\right\|^2 \\
&\le \frac{1}{2}\left\|\partial_z^l \sigma\right\|^2 + \frac{1}{2}\left\|(1-\Pi)\partial_z^l h\right\|_v^2.
\end{aligned}
\tag{5.6}
$$

For $i<l$,

$$
\begin{aligned}
-III \le& \frac{C_1}{2}\left\|\partial_z^i \sigma\right\|_{H^1(x,z)}\left(\left\|\partial_z^{l-i}\partial_x \phi\right\|^2 + \left\|\partial_z^l \partial_x \phi\right\|^2\right) \\
\le& \frac{C_1}{2}\left\|\partial_z^i h\right\|_{H^1(x,z)}\left(\left\|\partial_z^{l-i}\partial_x \phi\right\|^2 + \left\|\partial_z^l \partial_x \phi\right\|^2\right).
\end{aligned}
$$

For $i=l$,

$$
\begin{aligned}
-III =& \langle \partial_x \phi, \partial_z^l \sigma \partial_z^l \partial_x \phi\rangle = -\frac{1}{2}\langle \partial_x \phi, \partial_x (\partial_z^l \partial_x \phi)^2\rangle \le \frac{C_1}{2}\left\|\partial_x^2 \phi\right\|_{H^1(x,z)}\left\|\partial_z^l \partial_x \phi\right\|^2 \\
\le& \frac{C_1}{2}\left\|h\right\|_{H^1(x,z)}\left\|\partial_z^l \partial_x \phi\right\|^2.
\end{aligned}
\tag{5.7}
$$

Combining all terms in (5.1), one has,

$$
\begin{aligned}
&\delta\partial_t\left[\epsilon\langle \partial_z^l u, \partial_z^l \partial_x \phi\rangle + \frac{1}{2}\left\|\partial_z^l \partial_x \phi\right\|^2\right] + \frac{\epsilon}{2}\left\|\partial_z^l \sigma\right\|^2 + \delta\left\|\partial_z^l \partial_x \phi\right\|^2 \\
\le& \epsilon\left\|\partial_z^l u\right\|^2 + \frac{\epsilon}{2}\left\|(1-\Pi)\partial_z^l h\right\|_v^2 + \frac{C_1\delta}{2}\sum_{l\ne 0, i=0}^{l-1}\binom{l}{i}\left\|\partial_z^i h\right\|_{H^1(x,z)}\left(\left\|\partial_z^{l-i}\partial_x \phi\right\|^2 + \left\|\partial_z^l \partial_x \phi\right\|^2\right) \\
&+ \frac{C_1\delta}{2}\left\|h\right\|_{H^1(x,z)}\left\|\partial_z^l \partial_x \phi\right\|^2.
\end{aligned}
\tag{5.8}
$$

Summing l from 0 to m gives,

$$\delta\partial_t\left[\epsilon\sum_{l=0}^{m}\langle\partial_z^l u,\partial_z^l\partial_x\phi\rangle+\frac{1}{2}\|\partial_x\phi\|_{H^m}^2\right]+\frac{\epsilon}{2}\|\sigma\|_{H^m}^2+\delta\|\partial_x\phi\|_{H^m}^2$$
$$\leq\epsilon\|u\|_{H^m}^2+\frac{\epsilon}{2}\|(1-\Pi)h\|_{H_v^m}^2+2AC_1\delta\sqrt{E_h^m}\|\partial_x\phi\|_{H^m}^2. \tag{5.9}$$

5.2 Dissipation terms $\left\|\partial_z^l\partial_x\sigma\right\|^2$ and $\left\|\partial_z^l\partial_x^2\phi\right\|^2$

Taking ∂_z^l to (2.21) and multiplying by $\partial_z^l\partial_x\sigma$, then integrating it over $\mu(x,z)$,

$$\epsilon\delta\underbrace{\langle\partial_t\partial_z^l u,\partial_z^l\partial_x\sigma\rangle}_{I}+\epsilon\underbrace{\left\|\partial_z^l\partial_x\sigma\right\|^2}_{II.1}+\underbrace{\langle\partial_z^l u,\partial_z^l\partial_x\sigma\rangle}_{VI}+\delta\underbrace{\langle\partial_z^l\partial_x\phi,\partial_z^l\partial_x\sigma\rangle}_{V}$$
$$=-\epsilon\underbrace{\langle(1-\Pi)\partial_x\partial_z^l h,v^2\sqrt{M}\partial_z^l\partial_x\sigma\rangle}_{II.2}-\delta\sum_{i=0}^{l}\binom{l}{i}\underbrace{\langle\partial_z^{l-i}\partial_x\phi\partial_z^i\sigma,\partial_z^l\partial_x\sigma\rangle}_{III}. \tag{5.10}$$

Note that,

$$I=\partial_t\langle\partial_z^l u,\partial_x\partial_z^l\sigma\rangle-\langle\partial_z^l u,\partial_z^l\partial_x\partial_t\sigma\rangle=\partial_t\langle\partial_z^l\partial_x u,\partial_z^l\partial_x^2\phi\rangle-\frac{1}{\delta}\left\|\partial_z^l\partial_x u\right\|^2, \tag{5.11}$$

$$VI=\langle\partial_z^l u,\partial_z^l\partial_x\sigma\rangle=\delta\langle\partial_z^l\partial_t\sigma,\partial_z^l\sigma\rangle=\frac{\delta}{2}\partial_t\left\|\partial_z^l\sigma\right\|^2=\frac{\delta}{2}\partial_t\left\|\partial_z^l\partial_x^2\phi\right\|^2, \tag{5.12}$$

$$V=-\langle\partial_z^l\partial_x^2\phi,\partial_z^l\sigma\rangle=\left\|\partial_z^l\partial_x^2\phi\right\|^2, \tag{5.13}$$

$$-II.2\leq\frac{1}{2}\left\|\partial_z^l\partial_x\sigma\right\|^2+\frac{1}{2}\left\|(1-\Pi)\partial_z^l\partial_x h\right\|_v^2, \tag{5.14}$$

$$-III\leq\frac{C_1}{2}\left\|\partial_z^{l-i}\partial_x\phi\right\|_{H^1(x,z)}(\left\|\partial_z^i\sigma\right\|^2+\left\|\partial_z^l\partial_x\sigma\right\|^2). \tag{5.15}$$

Using (5.11) - (5.15) in (5.10) implies,

$$\delta\partial_t\left[\epsilon\langle\partial_z^l\partial_x u,\partial_z^l\partial_x^2\phi\rangle+\frac{1}{2}\left\|\partial_z^l\partial_x^2\phi\right\|^2\right]+\frac{\epsilon}{2}\left\|\partial_z^l\partial_x\sigma\right\|^2+\delta\left\|\partial_z^l\partial_x^2\phi\right\|^2$$
$$\leq\frac{\epsilon}{2}\left\|(1-\Pi)\partial_z^l\partial_x h\right\|_v^2+\epsilon\left\|\partial_z^l\partial_x u\right\|^2+\frac{C_1\delta}{2}\sum_{i=0}^{l}\binom{l}{i}\left\|\partial_z^{l-i}\partial_x\phi\right\|_{H^1(x,z)}(\left\|\partial_z^i\sigma\right\|^2+\left\|\partial_z^l\partial_x\sigma\right\|^2).$$

Summing l from 0 to $m-1$, one has,

$$\delta\partial_t\left[\epsilon\sum_{l=0}^{m-1}\langle\partial_z^l\partial_x u,\partial_z^l\partial_x^2\phi\rangle+\frac{1}{2}\left\|\partial_x^2\phi\right\|_{H^{m-1}}^2\right]+\frac{\epsilon}{2}\|\partial_x\sigma\|_{H^{m-1}}^2+\delta\left\|\partial_x^2\phi\right\|_{H^{m-1}}^2$$
$$\leq\epsilon\|\partial_x u\|_{H^{m-1}}^2+\frac{\epsilon}{2}\|(1-\Pi)\partial_x h\|_{H_v^{m-1}}^2+AC_1\delta\sqrt{E_\phi^m}(\|\partial_x\sigma\|_{H^{m-1}}^2+\|\sigma\|_{H^{m-1}}^2). \tag{5.16}$$

Combining (6.10) and (5.16), one finishes the energy estimates for the microscopic system,

$$\delta\partial_t\left[\underbrace{\epsilon\sum_{l=0}^{m-1}\langle\partial_z^l u,\partial_z^l\partial_x\phi\rangle+\epsilon\sum_{l=0}^{m}\langle\partial_z^l\partial_x u,\partial_z^l\partial_x^2\phi\rangle}_{I}+\underbrace{\frac{1}{2}E_\phi^m}_{VI}\right]+\frac{\epsilon}{2}\underbrace{D_\sigma^m}_{II.1}+\delta\underbrace{D_\phi^m}_{V}$$
$$\leq\epsilon\underbrace{D_u^m}_{I}+\frac{\epsilon}{2}\underbrace{D_h^m}_{II.2}+\underbrace{2AC_1\delta\sqrt{E_h^m}D_\phi^m+AC_1\delta\sqrt{E_\phi^m}D_\sigma^m}_{III}. \tag{5.17}$$

6 Exponential Decay to the Maxwellian

Before we do the analysis for the two energy estimates, we first go through the process in a more general framework. If one has the energy estimate,

$$\frac{1}{2}\partial_t \hat{E} + \alpha D \leq \beta\sqrt{\hat{E}}D, \tag{6.1}$$

and one wants to get an exponential decay for E, then one requires,

$$\text{REQUIREMENT 1:} \quad \hat{E} \sim E \leq D. \tag{6.2}$$

On the other hand, one needs the dissipations on the LHS to balance the "bad terms" on the RHS, so one requires,

$$\text{REQUIREMENT 2:} \quad \alpha > 0. \tag{6.3}$$

Since (6.1) is equivalent to,

$$\partial_t\sqrt{\hat{E}} \lesssim \frac{1}{\sqrt{\hat{E}}}\left(\beta\sqrt{\hat{E}} - \alpha\right) D, \tag{6.4}$$

therefore, if one assumes the initial data satisfies,

$$\frac{1}{\sqrt{\hat{E}}}\left(\beta\sqrt{\hat{E}} - \alpha\right) D \lesssim -\frac{\alpha}{\sqrt{\hat{E}}}D, \quad \text{or,} \quad \sqrt{\hat{E}(0)} \leq O\left(\frac{\alpha}{\beta}\right), \tag{6.5}$$

then by standard continuity argument, since $\sqrt{\hat{E}}$ is decreasing, so for $t > 0$,

$$\partial_t\sqrt{\hat{E}} \lesssim \frac{\alpha}{\sqrt{\hat{E}}}D, \tag{6.6}$$

and since $D \geq \hat{E}$, (6.6) implies the exponential decay,

$$\hat{E} \lesssim e^{-C\alpha t}\hat{E}(0) \quad \sim \quad E(t) \lesssim e^{-C\alpha t}E(0). \tag{6.7}$$

Furthermore, if one wants to get the optimal covergence rate with least restriction on initial data, then one needs,

$$\begin{aligned} &\text{REQUIREMENT 3:} \quad \sqrt{\hat{E}(0)} \leq O\left(\frac{\alpha}{\beta}\right) \text{ independent of small parameters.} \\ &\text{REQUIREMENT 4:} \quad \alpha \text{ should be as large as possible.} \end{aligned} \tag{6.8}$$

Remark 6.1. *Without uncertainty, if one directly uses the energy estimates from [15], then when the small parameter ϵ and δ are put in, the energy estimates become,*

$$\begin{aligned} &\frac{1}{2}\partial_t\left[E_h^m + \frac{1}{\epsilon}E_\phi^m\right] + \frac{1}{\epsilon}\left(D_h^m + D_u^m\right) \\ \lesssim &\frac{1}{\epsilon}\sqrt{E_h^m + E_\phi^m}(D_u^m + D_h^m + D_\phi^m + D_\sigma^m), \end{aligned} \tag{6.9}$$

and,

$$\begin{aligned} &\partial_t\left[\sum_{l=0}^{m-1}\langle\partial_z^l u, \partial_z^l\partial_x\phi\rangle + \sum_{l=0}^{m}\langle\partial_z^l\partial_x u, \partial_z^l\partial_x^2\phi\rangle + \frac{1}{2\epsilon}E_\phi^m\right] + \frac{1}{2}D_\sigma^m + \frac{1}{\epsilon}D_\phi^m \\ \lesssim &D_u^m + D_h^m + \frac{1}{\epsilon}\sqrt{E_h^m + E_\phi^m}(D_\phi^m + D_\sigma^m). \end{aligned} \tag{6.10}$$

Let $G^m = \sum_{l=0}^{m-1}\langle \partial_z^l u, \partial_z^l\partial_x\phi\rangle + \sum_{l=0}^{m}\langle \partial_z^l\partial_x u, \partial_z^l\partial_x^2\phi\rangle + \frac{1}{2\epsilon}E_\phi^m$. *Since* $-\epsilon E_h^m + \frac{1}{4\epsilon}E_\phi^m \le G^m \le \epsilon E_h^m + \frac{3}{4\epsilon}E_\phi^m$, *so if one combines the microscopic and macroscopic energy estimates (6.9) +* γ *(6.10), one needs* $\gamma \le O(\frac{1}{\epsilon})$ *to satisfy REQUIREMENT 1. Furthermore, if one wants to get the optimal convergence rate based on this energy estimate, then one needs the dissipation terms to be as large as possible, that is* γ *as large as possible, which means* $\gamma = O(\frac{1}{\epsilon})$. *Therefore one derives,*

$$\begin{aligned}&\frac{1}{2}\partial_t \hat{E}^m + \frac{1}{\epsilon}(D_h^m + D_u^m) + \frac{1}{\epsilon}D_\sigma^m + \frac{1}{\epsilon^2}D_\phi^m \\ \lesssim&\frac{1}{\epsilon}\sqrt{E_h^m + E_\phi^m}(D_u^m + D_h^m) + \frac{1}{\epsilon^2}\sqrt{E_h^m + E_\phi^m}D_\sigma^m + \frac{1}{\epsilon^2}\sqrt{E_h^m + E_\phi^m}D_\phi^m,\end{aligned} \tag{6.11}$$

where $\hat{E}^m = \left(E_h^m + \frac{1}{\epsilon}E_\phi^m\right) + \frac{1}{\epsilon}G^m \sim E_h^m + \frac{1}{\epsilon^2}E_\phi^m$. *So (6.11) leads to,*

$$\frac{1}{2}\partial_t \hat{E}^m + \frac{1}{\epsilon}(D_h^m + D_u^m) + \frac{1}{\epsilon}D_\sigma^m + \frac{1}{\epsilon^2}D_\phi^m \le \frac{1}{\epsilon}\sqrt{\hat{E}^m}(D_h^m + D_u^m) + \frac{1}{\epsilon^2}\sqrt{\hat{E}^m}D_\sigma^m + \frac{1}{\epsilon^2}\sqrt{\hat{E}^m}D_u^m. \tag{6.12}$$

Compare the term D_σ^m, *one notes that* $\sqrt{\hat{E}^m}$ *needs to be* $O(\epsilon)$ *such that the bad term on the RHS can be controlled by the* $O(1)$ *dissipation on the LHS. That is one requires*

$$E_h^m(0) + \frac{1}{\epsilon^2}E_\phi^m(0) \lesssim O(\epsilon) \tag{6.13}$$

to obtain the exponential decay

$$E_h^m + \frac{1}{\epsilon}E_\phi^m \lesssim e^{-O(1)t}\left(E_h^m(0) + \frac{1}{\epsilon}E_\phi^m\right). \tag{6.14}$$

This means the initial data $E_h^m = O(\epsilon)$, $E_\phi^m = O(\epsilon^3)$. *These conditions are much stronger than the one in (3.8) of Theorem 3.4.*

However, if the coefficient of the D_σ *only depends on* E_ϕ^m, *then (6.11) becomes,*

$$\begin{aligned}&\frac{1}{2}\partial_t \hat{E}^m + \frac{1}{\epsilon}(D_h^m + D_u^m) + \frac{1}{\epsilon}D_\sigma^m + \frac{1}{\epsilon^2}D_\phi^m \\ \lesssim&\frac{1}{\epsilon}\sqrt{E_h^m + E_\phi^m}(D_u^m + D_h^m) + \frac{1}{\epsilon}\sqrt{\frac{1}{\epsilon^2}E_\phi^m}D_\sigma^m + \frac{1}{\epsilon^2}\sqrt{E_h^m}D_\phi^m \\ \lesssim&\frac{1}{\epsilon}\sqrt{\hat{E}^m}(D_h^m + D_u^m) + \frac{1}{\epsilon}\sqrt{\hat{E}^m}D_\sigma^m + \frac{1}{\epsilon^2}\sqrt{\hat{E}^m}D_u^m.\end{aligned} \tag{6.15}$$

Now the bad terms and good terms can be well balanced even if the initial data of $\hat{E}^m$ *is* $O(1)$.

6.1 The high field regime

For the high field regime, where $\delta = 1$, set

$$\begin{aligned}&F^m = \epsilon E_h^m + E_\phi^m, \quad G^m = \epsilon\sum_{l=0}^{m-1}\langle \partial_z^l u, \partial_z^l\partial_x\phi\rangle + \epsilon\sum_{l=0}^{m}\langle \partial_z^l\partial_x u, \partial_z^l\partial_x^2\phi\rangle + \frac{1}{2}E_\phi^m, \\ &\hat{E}^m = F^m + \frac{2\lambda_0}{\epsilon}G^m, \quad E^m = \epsilon E_h^m + \frac{1}{\epsilon}E_\phi^m,\end{aligned} \tag{6.16}$$

where F^m is the term inside ∂_t in (3.5) and G^m is that in (3.6).

By (2.20) and Young's Inequality, one can bound G^m by

$$\begin{aligned}-\epsilon^2 E_h^m + (\frac{1}{2} - \frac{1}{4})E_\phi^m \leq & G^m \leq \epsilon^2 E_h^m + (\frac{1}{2} + \frac{1}{4})E_\phi^m, \\ -\epsilon^2 E_h^m + \frac{1}{4}E_\psi^m \leq & G^m \leq \epsilon^2 E_h^m + \frac{3}{4}E_\psi^m.\end{aligned} \tag{6.17}$$

Since $\lambda_0 \leq \frac{1}{4}$, thus one obtains,

$$\begin{aligned}(1-2\lambda_0)\epsilon E_h^m + (\epsilon + \frac{\lambda_0}{2})\frac{1}{\epsilon}E_\phi^m \leq & \hat{E}^m \leq (1+2\lambda_0)\,\epsilon E_h^m + \left(\epsilon + \frac{3\lambda_0}{2}\right)\frac{1}{\epsilon}E_\phi^m, \\ \frac{\lambda_0}{2}E^m \leq & \hat{E}^m \leq \frac{3}{2}E^m,\end{aligned} \tag{6.18}$$

or equivalently,

$$\frac{3}{2}\sqrt{\hat{E}^m} \leq \sqrt{\epsilon E_h^m} + \sqrt{\frac{1}{\epsilon}E_\phi^m} \leq \frac{2}{\lambda_0}\sqrt{\hat{E}^m}. \tag{6.19}$$

So one has the equivalence between the energies E^m and $\hat{E}^m$,

$$\hat{E}^m \sim E^m \leq \epsilon\,(D_u^m + D_h^m + D_\sigma^m) + \frac{1}{\epsilon}D_\phi^m. \tag{6.20}$$

By (3.5) + $\frac{\lambda_0}{\epsilon}$(3.6), one has the energy estimates,

$$\begin{aligned}&\frac{1}{2}\partial_t \hat{E}^m + \left(\lambda_0 - \frac{\lambda_0}{2}\right)D_h^m + (1-\lambda_0)\,D_u^m + \frac{\lambda_0}{2}D_\sigma^m + \frac{\lambda_0}{\epsilon}D_\phi^m \\ \leq & AC_1\left[\left(\frac{1}{\sqrt{\epsilon}}\sqrt{\epsilon E_h^m} + 2\sqrt{\epsilon}\sqrt{\frac{1}{\epsilon}E_\phi^m}\right)(5D_u^m + 4D_h^m) + \sqrt{\epsilon}\left(1+\frac{1}{\epsilon}\right)\sqrt{\frac{1}{\epsilon}E_\phi^m}D_\sigma^m + \frac{1}{\sqrt{\epsilon}}\left(1+\frac{1}{\epsilon}\right)\sqrt{\epsilon E_h^m}D_\phi^m\right] \\ \leq & \frac{10}{\sqrt{\epsilon}}AC_1\left(\frac{2}{\lambda_0}\sqrt{\hat{E}^m}\right)(D_h^m + D_u^m) + \frac{2}{\sqrt{\epsilon}}AC_1\left(\frac{2}{\lambda_0}\sqrt{\hat{E}^m}\right)D_\sigma^m + \frac{2}{\epsilon^{3/2}}AC_1\left(\frac{2}{\lambda_0}\sqrt{\hat{E}^m}\right)D_\phi^m,\end{aligned} \tag{6.21}$$

which implies

$$\begin{aligned}&\frac{1}{2}\partial_t \hat{E}^m + \frac{\lambda_0}{2}(D_h^m + D_u^m) + \frac{\lambda_0}{2}D_\sigma^m + \frac{\lambda_0}{\epsilon}D_\phi^m \\ \leq & \frac{20AC_1}{\lambda_0\sqrt{\epsilon}}\sqrt{\hat{E}^m}(D_h^m + D_u^m) + \frac{4AC_1}{\lambda_0\sqrt{\epsilon}}\sqrt{\hat{E}^m}D_\sigma^m + \frac{4AC_1}{\lambda_0\epsilon^{3/2}}\sqrt{\hat{E}^m}D_\psi^m.\end{aligned} \tag{6.22}$$

Therefore, by standard continuity argument, under the condition of,

$$\sqrt{\hat{E}^m(0)} \leq \min\left\{\frac{\frac{\lambda_0}{4}}{\frac{20AC_1}{\lambda_0\epsilon^{1/2}}}, \frac{\frac{\lambda_0}{4}}{\frac{4AC_1}{\lambda_0\epsilon^{1/2}}}, \frac{\frac{\lambda_0}{2\epsilon}}{\frac{4AC_1}{\lambda_0\epsilon^{3/2}}}\right\},$$

which holds if,

$$\hat{E}^m(0) \leq \left(\frac{\lambda_0^2\epsilon^{1/2}}{80AC_1}\right)^2,$$

or equivalently,

$$E_h^m(0) + \frac{1}{\epsilon^2}E_\phi^m(0) \leq \frac{2\lambda_0^3}{(80AC_1)^2}, \tag{6.23}$$

one then has the estimate,

$$\frac{1}{2}\partial_t \hat{E}^m + \frac{\lambda_0}{4}(D_h^m + D_u^m + D_\sigma^m + \frac{1}{\epsilon}D_\phi^m) \leq 0. \tag{6.24}$$

Then by (6.18) and (6.20), one ends up with the energy estimate,

$$\frac{1}{2}\partial_t \hat{E}^m + \frac{\lambda_0}{6}\hat{E}^m \leq 0 \tag{6.25}$$

which gives exponential convergence rate for the energy,

$$\hat{E}^m(t) \leq e^{-\frac{\lambda_0}{3}t}\hat{E}^m(0).,\tag{6.26}$$

or equivalently,

$$E_h^m(t) + \frac{1}{\epsilon^2}E_\phi^m(t) \leq \frac{3}{\lambda_0}e^{-\frac{\lambda_0}{3}t}(E_h^m(0) + \frac{1}{\epsilon^2}E_\phi^m(0)).\tag{6.27}$$

This completes the proof of (3.9) in Theorem 3.4.

6.2 The parabolic regime

For the parabolic regime, where $\delta = \epsilon$, set

$$F^m = \epsilon E_h^m + \epsilon E_\phi^m, \quad G^m = \epsilon\sum_{l=0}^{m-1}\langle \partial_z^l u, \partial_z^l\partial_x\phi\rangle + \epsilon\sum_{l=0}^{m}\langle \partial_z^l\partial_x u, \partial_z^l\partial_x^2\phi\rangle + \frac{1}{2}E_\phi^m,\tag{6.28}$$

$$\hat{E}^m = F^m + 2\lambda_0 G^m, \quad E^m = \epsilon E_h^m + E_\phi^m.\tag{6.29}$$

Similar to (6.17), the bounds of G^m is,

$$-\epsilon E_h^m + \frac{1}{4}E_\phi^m \leq -\epsilon E_h^m + \left(\frac{1}{2} - \frac{\epsilon}{4}\right)E_\phi^m \leq G^m \leq \epsilon E_h^m + \left(\frac{1}{2} + \frac{\epsilon}{4}\right)E_\phi^m \leq \epsilon E_h^m + \frac{3}{4}E_\phi^m,\tag{6.30}$$

and $\lambda_0 \leq \frac{1}{4}$, so one obtains,

$$\begin{aligned}(1-2\lambda_0)\epsilon E_h^m + (\epsilon + \frac{\lambda_0}{2})E_\phi^m \leq &\hat{E}^m \leq (1+2\lambda_0)\epsilon E_h^m + (\epsilon + \frac{3\lambda_0}{2}),\\ \frac{\lambda_0}{2}E^m \leq &\hat{E}^m \leq \frac{3}{2}E^m,\end{aligned}\tag{6.31}$$

or equivalently,

$$\frac{2}{3}\sqrt{\hat{E}^m} \leq \epsilon^{1/2}\sqrt{E_h^m} + \sqrt{E_\phi^m} \leq \frac{2}{\lambda_0}\sqrt{\hat{E}^m}.\tag{6.32}$$

By (3.5) + λ_0(3.6), one has energy estimates,

$$\begin{aligned}&\frac{1}{2}\partial_t\hat{E}^m + \left(\frac{\lambda_0}{\epsilon} - \frac{\lambda_0}{2}\right)D_h^m + \left(\frac{1}{\epsilon} - \lambda_0\right)D_u^m + \frac{\lambda_0}{2}D_\sigma^m + \lambda_0 D_\phi^m\\ \leq &AC_1\left(\sqrt{E_h^m} + 2\sqrt{E_\phi^m}\right)(5D_u^m + 4D_h^m) + AC_1(1+\lambda_0)\sqrt{E_\phi^m}D_\sigma^m + AC_1(1+2\lambda_0)\sqrt{E_h^m}D_\phi^m\\ \leq &10AC_1\left(\frac{1}{\epsilon^{1/2}}\frac{2}{\lambda_0}\sqrt{\hat{E}^m}\right)(D_u^m + D_h^m) + 2AC_1\left(\frac{2}{\lambda_0}\sqrt{\hat{E}^m}\right)D_\sigma^m + 2AC_1\left(\frac{1}{\epsilon^{1/2}}\frac{2}{\lambda_0}\sqrt{\hat{E}^m}\right)D_\phi^m,\end{aligned}\tag{6.33}$$

which implies,

$$
\begin{aligned}
&\frac{1}{2}\partial_t\hat{E}^m + \frac{\lambda_0}{2\epsilon^2}\left(\epsilon D_h^m + \epsilon D_u^m)\right) + \frac{\lambda_0}{2\epsilon^2}\left(\epsilon D_\sigma^m\right) + \lambda_0 D_\phi^m \\
\leq &\frac{20AC_1}{\lambda_0\epsilon^{3/2}}\sqrt{\hat{E}^m}\left(\epsilon D_h^m + \epsilon D_u^m)\right) + \frac{4AC_1}{\lambda_0\epsilon}\sqrt{\hat{E}^m}\left(\epsilon D_\sigma^m\right) + \frac{4AC_1}{\lambda_0\epsilon^{1/2}}\sqrt{\hat{E}^m}D_\phi^m. \qquad (6.34)
\end{aligned}
$$

So if the initial data satisfies the condition

$$
\begin{aligned}
\sqrt{\hat{E}^m(0)} &\leq \min\left\{\frac{\frac{\lambda_0}{4\epsilon^2}}{\frac{20AC_1}{\lambda_0\epsilon^{3/2}}}, \frac{\frac{\lambda_0}{4\epsilon^2}}{\frac{4AC_1}{\lambda_0\epsilon}}, \frac{\frac{\lambda_0}{2}}{\frac{4AC_1}{\lambda_0\epsilon^{1/2}}}\right\} \\
\hat{E}^m(0) &\leq \left(\frac{\lambda_0^2\epsilon^{1/2}}{80AC_1}\right)^2, \\
\text{or equivalently,}\qquad E_h^m(0) + \frac{1}{\epsilon}E_\phi^m(0) &\leq \frac{2\lambda_0^3}{(80AC_1)^2}, \qquad (6.35)
\end{aligned}
$$

the similar to (6.24) - (6.26), one has the energy decay,

$$
\hat{E}^m(t) \leq e^{-\frac{\lambda_0}{2}t}\hat{E}^m(0), \qquad (6.36)
$$

or equivalently,

$$
E_h^m(t) + \frac{1}{\epsilon}E_\phi^m(t) \leq \frac{3}{\lambda_0}e^{-\frac{\lambda_0}{2}t}(E_h^m(0) + \frac{1}{\epsilon}E_\phi^m(0)). \qquad (6.37)
$$

This completes the proof of (3.11) in Theorem 3.4.

Appendix: The proof of Lemma 3.8

Proof. (a) By the definition of u in (2.14), and (2.16), (2.21),

$$
\begin{aligned}
\left\langle\partial^{\mathbf{k}}\partial_x\phi, v\sqrt{M}\partial^{\mathbf{k}}h\right\rangle &= \left\langle\partial^{\mathbf{k}}\partial_x\phi, \partial^{\mathbf{k}}u\right\rangle = -\left\langle\partial^{\mathbf{k}}\phi, \partial^{\mathbf{k}}\partial_x u\right\rangle = \delta\left\langle\partial^{\mathbf{k}}\phi, \partial^{\mathbf{k}}\partial_t\sigma\right\rangle \\
= -\,\delta\left\langle\partial^{\mathbf{k}}\phi, \partial^{\mathbf{k}}\partial_x^2\partial_t\phi\right\rangle &= \delta\left\langle\partial^{\mathbf{k}}\partial_x\phi, \partial^{\mathbf{k}}\partial_x\partial_t\phi\right\rangle = \frac{\delta}{2}\partial_t\left\|\partial^{\mathbf{k}}\partial_x\phi\right\|^2, \qquad (A.1)
\end{aligned}
$$

where the last equality of the first line is because of (2.21), and the first equality of the second line is because of (2.16).

(b) First break $\partial^{\mathbf{k}}h = \partial^{\mathbf{k}}\sigma\sqrt{M} + (\partial^{\mathbf{k}}h - \partial^{\mathbf{k}}\sigma\sqrt{M})$, and then use $\partial^{\mathbf{k}}h - \partial^{\mathbf{k}}\sigma\sqrt{M} = \partial^{\mathbf{k}}u\,v\sqrt{M} + (1-\Pi)\partial^{\mathbf{k}}h$, one has,

$$
\begin{aligned}
&\left\langle\partial^{\mathbf{k}}\partial_x\phi\,\partial_v(\partial^{\mathbf{i}}h), \partial^{\mathbf{l}}h\right\rangle \\
=&\left\langle\partial^{\mathbf{k}}\partial_x\phi\,\partial_v(\partial^{\mathbf{i}}\sigma\sqrt{M}), \partial^{\mathbf{l}}h\right\rangle + \left\langle\partial^{\mathbf{k}}\partial_x\phi\,\partial_v(\partial^{\mathbf{i}}h - \partial^{\mathbf{i}}\sigma\sqrt{M}), \partial^{\mathbf{l}}\sigma\sqrt{M} + \left(\partial^{\mathbf{l}}h - \partial^{\mathbf{l}}\sigma\sqrt{M}\right)\right\rangle \\
=&-\frac{1}{2}\left\langle\partial^{\mathbf{k}}\partial_x\phi\,\partial^{\mathbf{i}}\sigma, \partial^{\mathbf{l}}hv\sqrt{M}\right\rangle + \left\langle\partial^{\mathbf{k}}\partial_x\phi\,\partial_v(\partial^{\mathbf{i}}h), \partial^{\mathbf{l}}\sigma\sqrt{M}\right\rangle - \left\langle\partial^{\mathbf{k}}\partial_x\phi\,\partial_v(\partial^{\mathbf{i}}\sigma\sqrt{M}), \partial^{\mathbf{l}}\sigma\sqrt{M}\right\rangle \\
&+\left\langle\partial^{\mathbf{k}}\partial_x\phi, \partial_v\left(\partial^{\mathbf{i}}uv\sqrt{M} + (1-\Pi)\partial^{\mathbf{i}}h\right)\left(\partial^{\mathbf{l}}uv\sqrt{M} + (1-\Pi)\partial^{\mathbf{l}}h\right)\right\rangle \\
\leq&-\frac{1}{2}\left\langle\partial^{\mathbf{k}}\partial_x\phi, \partial^{\mathbf{i}}\sigma\partial^{\mathbf{l}}u\right\rangle - \left\langle\partial^{\mathbf{k}}\partial_x\phi\,\partial^{\mathbf{i}}h, \partial^{\mathbf{l}}\sigma\partial_v(\sqrt{M})\right\rangle + \frac{1}{2}\left\langle\partial^{\mathbf{k}}\partial_x\phi, \partial^{\mathbf{i}}\sigma\partial^{\mathbf{l}}\sigma vM\right\rangle \\
&+\left\|\partial^{\mathbf{k}}\partial_x\phi\right\|_{L^\infty(x,z)}(\left\|\partial^{\mathbf{i}}u\partial_v(v\sqrt{M})\right\|^2 + \left\|\partial_v(1-\Pi)\partial^{\mathbf{i}}h\right\|^2 + \left\|\partial^{\mathbf{l}}u(v\sqrt{M})\right\|^2 + \left\|(1-\Pi)\partial^{\mathbf{l}}h\right\|^2) \\
\leq&-\frac{1}{2}\left\langle\partial^{\mathbf{k}}\partial_x\phi, \partial^{\mathbf{i}}\sigma\partial^{\mathbf{l}}u\right\rangle + \frac{1}{2}\left\langle\partial^{\mathbf{k}}\partial_x\phi, \partial^{\mathbf{l}}\sigma\partial^{\mathbf{i}}u\right\rangle + 0 \\
&+C_1\left\|\partial^{\mathbf{k}}\partial_x\phi\right\|_{H^1(x,z)}(\frac{3}{4}\left\|\partial^{\mathbf{i}}u\right\|^2 + \left\|(1-\Pi)\partial^{\mathbf{i}}h\right\|_v^2 + \left\|\partial^{\mathbf{l}}u\right\|^2 + \left\|(1-\Pi)\partial^{\mathbf{l}}h\right\|_v^2), \qquad (A.2)
\end{aligned}
$$

where the last inequality comes from the Sobolev embedding for 1D,

$$\|f\|_{C^{k-1}(x,z)} \leq C_k \|f\|_{H^k(x,z)}, \quad \text{for } \forall f \in H^k(x,z), \tag{A.3}$$

where C_k is a constant only depending on k.

Next,

$$\begin{aligned}
&-\frac{1}{2}\left\langle v\partial^{\mathbf{k}}\partial_x\phi\,\partial^{\mathbf{i}}h, \partial^{\mathbf{l}}h\right\rangle \\
=&-\frac{1}{2}\left\langle v\partial^{\mathbf{k}}\partial_x\phi\,\partial^{\mathbf{i}}\sigma\sqrt{M}, \partial^{\mathbf{l}}h\right\rangle - \frac{1}{2}\left\langle v\partial^{\mathbf{k}}\partial_x\phi\left(\partial^{\mathbf{i}}h - \partial^{\mathbf{i}}\sigma\sqrt{M}\right), \partial^{\mathbf{l}}\sigma\sqrt{M} + \left(\partial^{\mathbf{l}}h - \partial^{\mathbf{l}}\sigma\sqrt{M}\right)\right\rangle \\
=&-\frac{1}{2}\left\langle \partial^{\mathbf{k}}\partial_x\phi, \partial^{\mathbf{i}}\sigma\partial^{\mathbf{l}}u\right\rangle - \frac{1}{2}\left\langle \partial^{\mathbf{k}}\partial_x\phi, v\partial^{\mathbf{i}}h\,\partial^{\mathbf{l}}\sigma\sqrt{M}\right\rangle + \frac{1}{2}\left\langle \partial^{\mathbf{k}}\partial_x\phi, v\partial^{\mathbf{i}}\sigma\sqrt{M}\,\partial^{\mathbf{l}}\sigma\sqrt{M}\right\rangle \\
&-\frac{1}{2}\left\langle \partial^{\mathbf{k}}\partial_x\phi, v\left(\partial^{\mathbf{i}}uv\sqrt{M} + (1-\Pi)\partial^{\mathbf{i}}h\right)\left(\partial^{\mathbf{l}}uv\sqrt{M} + (1-\Pi)\partial^{\mathbf{l}}h\right)\right\rangle \\
\leq&-\frac{1}{2}\left\langle \partial^{\mathbf{k}}\partial_x\phi, \partial^{\mathbf{i}}\sigma\partial^{\mathbf{l}}u\right\rangle - \frac{1}{2}\left\langle \partial^{\mathbf{k}}\partial_x\phi, \partial^{\mathbf{i}}u\partial^{\mathbf{l}}\sigma\right\rangle + 0 \\
&+\frac{1}{2}\left\|\partial^{\mathbf{k}}\partial_x\phi\right\|_{L^\infty(x,z)}\left(\int |v|(\partial^{\mathbf{i}}uv\sqrt{M})^2 d\mu + \int |v|\left((1-\Pi)\partial^{\mathbf{i}}h\right)^2 d\mu\right. \\
&\left.+\int |v|(\partial^{\mathbf{l}}uv\sqrt{M})^2 d\mu + \int |v|((1-\Pi)\partial^{\mathbf{l}}h)^2 d\mu\right) \\
\leq&-\frac{1}{2}\left\langle \partial^{\mathbf{k}}\partial_x\phi, \partial^{\mathbf{i}}\sigma\partial^{\mathbf{l}}u\right\rangle - \frac{1}{2}\left\langle \partial^{\mathbf{k}}\partial_x\phi, \partial^{\mathbf{i}}u\partial^{\mathbf{l}}\sigma\right\rangle \\
&+\frac{1}{2}C_1\left\|\partial^{\mathbf{k}}\partial_x\phi\right\|_{H^1(x,z)}\left(2\left\|\partial^{\mathbf{i}}u\right\|^2 + \frac{1}{2}\left\|(1-\Pi)\partial^{\mathbf{i}}h\right\|_v^2 + 2\left\|\partial^{\mathbf{l}}u\right\|^2 + \frac{1}{2}\left\|(1-P)\partial^{\mathbf{l}}h\right\|_v^2\right).
\end{aligned} \tag{A.4}$$

Therefore (A.39) + (A.41) gives,

$$\begin{aligned}
&\left\langle \partial^{\mathbf{k}}\partial_x\phi\,\partial_v(\partial^{\mathbf{i}}h), \partial^{\mathbf{l}}h\right\rangle - \frac{1}{2}\left\langle v\partial^{\mathbf{k}}\partial_x\phi\,\partial^{\mathbf{i}}h, \partial^{\mathbf{l}}h\right\rangle \\
\leq&-\left\langle \partial^{\mathbf{k}}\partial_x\phi, \partial^{\mathbf{i}}\sigma\partial^{\mathbf{l}}u\right\rangle + C_1\left\|\partial^{\mathbf{k}}\partial_x\phi\right\|_{H^1(x,z)}\left(2\left\|\partial^{\mathbf{i}}u\right\|^2 + 2\left\|(1-\Pi)\partial^{\mathbf{i}}h\right\|_v^2 + 2\left\|\partial^{\mathbf{l}}u\right\|^2 + 2\left\|(1-P)\partial^{\mathbf{l}}h\right\|_v^2\right) \\
\leq&C_1\left\|\partial^{\mathbf{k}}\partial_x\phi\right\|_{H^1(x,z)}\left(\left\|\partial^{\mathbf{i}}\sigma\right\|^2 + 2\left\|\partial^{\mathbf{i}}u\right\|^2 + 2\left\|(1-\Pi)\partial^{\mathbf{i}}h\right\|_v^2 + 3\left\|\partial^{\mathbf{l}}u\right\|^2 + 2\left\|(1-P)\partial^{\mathbf{l}}h\right\|_v^2\right).
\end{aligned}$$

(c) Since

$$\begin{aligned}
&\left\langle \partial^{\mathbf{k}}\partial_x\phi\,\partial_v(\partial^{\mathbf{i}}h), \partial^{\mathbf{l}}h\right\rangle \\
=&\left\langle \partial^{\mathbf{k}}\partial_x\phi\,\partial_v(\partial^{\mathbf{i}}h), \partial^{\mathbf{l}}\sigma\sqrt{M}\right\rangle + \left\langle \partial^{\mathbf{k}}\partial_x\phi\,\partial_v(\partial^{\mathbf{i}}h), \partial^{\mathbf{l}}h - \partial^{\mathbf{l}}\sigma\sqrt{M}\right\rangle \\
=&-\left\langle \partial^{\mathbf{k}}\partial_x\phi\,\partial^{\mathbf{i}}h, \partial^{\mathbf{l}}\sigma\partial_v(\sqrt{M})\right\rangle - \left\langle \partial^{\mathbf{k}}\partial_x\phi\,\partial^{\mathbf{i}}h, \partial_v(\partial^{\mathbf{l}}uv\sqrt{M} + (1-\Pi)\partial^{\mathbf{l}}h)\right\rangle \\
\leq&\frac{1}{2}\left\langle \partial^{\mathbf{k}}\partial_x\phi, \partial^{\mathbf{i}}u\partial^{\mathbf{l}}\sigma\right\rangle + \int \left|\partial^{\mathbf{k}}\partial_x\phi\right| \left\|\partial^{\mathbf{i}}h\right\|_{L^2(v)} \left\|\partial^{\mathbf{l}}u\partial_v(v\sqrt{M}) + \partial_v(1-\Pi)\partial^{\mathbf{l}}h\right\|_{L^2(v)} d\mu(x,z) \\
\leq&\frac{1}{2}\langle \partial^{\mathbf{k}}\partial_x\phi, \partial^{\mathbf{l}}\sigma\partial^{\mathbf{i}}u\rangle + \left\|\left\|\partial^{\mathbf{i}}h\right\|_{L^2(v)}\right\|_{L^\infty(x,z)}\left(\frac{1}{2}\left\|\partial^{\mathbf{k}}\partial_x\phi\right\|^2 + \left\|\partial^{\mathbf{l}}u\partial_v(v\sqrt{M})\right\|^2 + \left\|\partial_v(1-\Pi)\partial^{\mathbf{l}}h\right\|^2\right) \\
\leq&\frac{1}{2}\langle \partial^{\mathbf{k}}\partial_x\phi, \partial^{\mathbf{l}}\sigma\partial^{\mathbf{i}}u\rangle + C_1\left\|\partial^{\mathbf{i}}h\right\|_{H^1(x,z)}\left(\frac{1}{2}\left\|\partial^{\mathbf{k}}\partial_x\phi\right\|^2 + \frac{3}{4}\left\|\partial^{\mathbf{l}}u\right\|^2 + \left\|(1-\Pi)\partial^{\mathbf{l}}h\right\|_v^2\right),
\end{aligned} \tag{A.5}$$

where the last inequality comes from

$$
\begin{aligned}
&\left\| \|f\|_{L^2(v)} \right\|_{L^\infty(x,z)}^2 \le C_1 \left\| \|f\|_{L^2(v)} \right\|_{H^1(x,z)}^2 \\
=& \int_{(0,l)\times I_z} \int_{\mathbb{R}} f^2 dv\, d\mu + \int_{(0,l)\times I_z} \left(\partial_x \left(\int_{\mathbb{R}} f^2\, dv \right)^{\frac{1}{2}} \right)^2 d\mu + \int_{(0,l)\times I_z} \left(\partial_z \left(\int f^2\, dv \right)^{\frac{1}{2}} \right)^2 d\mu \\
=& \|f\|_{L^2} + \int_{(0,l)\times I_z} \left(\frac{\int_{\mathbb{R}} f \partial_x f\, dv}{\left(\int_{\mathbb{R}} f^2\, dv\right)^{\frac{1}{2}}} \right)^2 d\mu + \int_{(0,l)\times I_z} \left(\frac{\int_{\mathbb{R}} f \partial_z f\, dv}{\left(\int_{\mathbb{R}} f^2\, dv\right)^{\frac{1}{2}}} \right)^2 d\mu \\
\le& \|f\|_{L^2} + \int_{(0,l)\times I_z} \left(\frac{\|f\|_{L^2(v)} \|\partial_x f\|_{L^2(v)}}{\|f\|_{L^2(v)}} \right)^2 d\mu + \int_{(0,l)\times I_z} \left(\frac{\|f\|_{L^2(v)} \|\partial_x f\|_{L^2(v)}}{\|f\|_{L^2(v)}} \right)^2 d\mu \\
=& \|f\|_{L^2} + \|\partial_x f\|^2 + \|\partial_z f\|^2 = \|f\|_{H^1(x,z)}^2 . \qquad (A.6)
\end{aligned}
$$

Next, similar to (A.42),

$$
\begin{aligned}
&-\frac{1}{2} \left\langle v \partial^{\mathbf{k}} \partial_x \phi\, \partial^{\mathbf{i}} h, \partial^{\mathbf{l}} h \right\rangle = -\frac{1}{2} \left\langle \partial^{\mathbf{k}} \partial_x \phi, v \partial^{\mathbf{i}} h\, \partial^{\mathbf{l}} h \right\rangle \\
=& -\frac{1}{2} \left\langle \partial^{\mathbf{k}} \partial_x \phi, v \partial^{\mathbf{i}} h\, \partial^{\mathbf{l}} \sigma \sqrt{M} \right\rangle - \frac{1}{2} \left\langle \partial^{\mathbf{k}} \partial_x \phi, v \partial^{\mathbf{i}} h \left(\partial^{\mathbf{l}} u v \sqrt{M} + (1-\Pi) \partial^{\mathbf{l}} h \right) \right\rangle \\
\le& -\frac{1}{2} \langle \partial^{\mathbf{k}} \partial_x \phi, \partial^{\mathbf{l}} \sigma \partial^{\mathbf{i}} u \rangle + \frac{1}{2} \int |\partial^{\mathbf{k}} \partial_x \phi| \left\| \partial^{\mathbf{i}} h \right\|_{L^2(v)} \left\| \partial^{\mathbf{l}} u v^2 \sqrt{M} + v(1-\Pi) \partial^{\mathbf{l}} h \right\|_{L^2(v)} d\mu(x,z) \\
\le& -\frac{1}{2} \langle \partial^{\mathbf{k}} \partial_x \phi, \partial^{\mathbf{l}} \sigma \partial^{\mathbf{i}} u \rangle + \frac{1}{2} \left\| \left\| \partial^{\mathbf{i}} h \right\|_{L^2(v)} \right\|_{L^\infty(x,z)} \left(\frac{1}{2} \left\| \partial^{\mathbf{k}} \partial_x \phi \right\|^2 + \left\| \partial^{\mathbf{l}} u (v^2 \sqrt{M}) \right\|^2 + \left\| v(1-\Pi) \partial^{\mathbf{l}} h \right\|^2 \right) \\
\le& -\frac{1}{2} \langle \partial^{\mathbf{k}} \partial_x \phi, \partial^{\mathbf{l}} \sigma \partial^{\mathbf{i}} u \rangle + \frac{C_1}{2} \left\| \partial^{\mathbf{i}} h \right\|_{H^1(x,z)} \left(\frac{1}{2} \left\| \partial^{\mathbf{k}} \partial_x \phi \right\|^2 + 3 \left\| \partial^{\mathbf{l}} u \right\|^2 + \left\| (1-\Pi) \partial^{\mathbf{l}} h \right\|_v^2 \right).
\end{aligned}
\qquad (A.7)
$$

Therefore (A.42) + (A.44) gives,

$$
\begin{aligned}
&\left\langle \partial^{\mathbf{k}} \partial_x \phi\, \partial_v (\partial^{\mathbf{i}} h), \partial^{\mathbf{l}} h \right\rangle - \frac{1}{2} \left\langle v \partial^{\mathbf{k}} \partial_x \phi\, \partial^{\mathbf{i}} h, \partial^{\mathbf{l}} h \right\rangle \\
\le& C_1 \left\| \partial^{\mathbf{i}} h \right\|_{H^1(x,z)} \left(\left\| \partial^{\mathbf{k}} \partial_x \phi \right\|^2 + 3 \left\| \partial^{\mathbf{l}} u \right\|^2 + 2 \left\| (1-\Pi) \partial^{\mathbf{l}} h \right\|_v^2 \right).
\end{aligned}
\qquad (A.8)
$$

(d) By (2.21) and (2.15) one derives,

$$
\partial_x (\partial^{\mathbf{k}} \partial_x \partial_t \phi) = -\partial^{\mathbf{k}} \partial_t \sigma = \partial_x \left(\frac{1}{\delta} \partial^{\mathbf{k}} u\right), \qquad (A.9)
$$

which implies,

$$
\partial^{\mathbf{k}} \partial_x \partial_t \phi(x) - \partial^{\mathbf{k}} \partial_x \partial_t \phi(0) = \frac{1}{\delta} \partial^{\mathbf{k}} u(x) - \frac{1}{\delta} \partial^{\mathbf{k}} u(0), \qquad (A.10)
$$

$$
\int_0^l \left(\partial^{\mathbf{k}} \partial_x \partial_t \phi(x)\, dx - \frac{1}{\delta} \partial^{\mathbf{k}} u(x) \right) dx = \left(\partial^{\mathbf{k}} \partial_x \partial_t \phi(0) - \frac{1}{\delta} \partial^{\mathbf{k}} u(0) \right) l. \qquad (A.11)
$$

Because of the periodic condition for ϕ, $\int_0^l \partial^{\mathbf{k}} \partial_x \phi dx = 0$, then one obtains,

$$
\partial^{\mathbf{k}} \partial_x \partial_t \phi = \frac{1}{\delta} \partial^{\mathbf{k}} u + \left(\partial^{\mathbf{k}} \partial_x \partial_t \phi(0) - \frac{1}{\delta} \partial^{\mathbf{k}} u(0) \right) = \frac{1}{\delta} \partial^{\mathbf{k}} u - \frac{1}{l\delta} \int_0^l \partial^{\mathbf{k}} u(x)\, dx. \qquad (A.12)
$$

Hence,

$$\begin{aligned}\left\|\partial^{\mathbf{k}}\partial_x\partial_t\phi\right\|^2 =& \frac{1}{\delta^2}\left\|\partial^{\mathbf{k}}u\right\|^2 + \frac{1}{\delta^2 l^2}\int_{I_z}\int_0^l\left(\int_0^l \partial^{\mathbf{k}}u\,dx\right)^2 d\mu(x,z) - \frac{2}{l\delta^2}\int_{I_z}\left(\int_0^l \partial^{\mathbf{k}}u\,dx\right)^2 d\mu(z)\\ =& \frac{1}{\delta^2}\left\|\partial^{\mathbf{k}}u\right\|^2 - \frac{1}{\delta^2 l}\int_{I_z}\left(\int_0^l \partial^{\mathbf{k}}u\,dx\right)^2 d\mu(z) \le \frac{1}{\delta^2}\left\|\partial^{\mathbf{k}}u\right\|^2. \qquad \text{(A.13)}\end{aligned}$$

■

References

[1] Anton Arnold, José A Carrillo, Irene Gamba, and C-W Shu. Low and high field scaling limits for the vlasov–and wigner–poisson–fokker–planck systems. *Transport theory and statistical physics*, 30(2-3):121–153, 2001.

[2] Ivo Babuska, Raúl Tempone, and Georgios E Zouraris. Galerkin finite element approximations of stochastic elliptic partial differential equations. *SIAM Journal on Numerical Analysis*, 42(2):800–825, 2004.

[3] Michal Branicki and Andrew J Majda. Fundamental limitations of polynomial chaos for uncertainty quantification in systems with intermittent instabilities. *Comm. Math. Sci*, 11(1):55–103, 2013.

[4] Subrahmanyan Chandrasekhar. Stochastic probems in physics and astronomy. *Rev. Modern Phys.*, 15:1–89, 19432013.

[5] Albert Cohen, Ronald DeVore, and Christoph Schwab. Convergence rates of best n-term galerkin approximations for a class of elliptic spdes. *Foundations of Computational Mathematics*, 10(6):615–646, 2010.

[6] Albert Cohen, Ronald Devore, and Christoph Schwab. Analytic regularity and polynomial approximation of parametric and stochastic elliptic pde's. *Analysis and Applications*, 9(01):11–47, 2011.

[7] Bruno Després and Benoît Perthame. Uncertainty propagation; intrusive kinetic formulations of scalar conservation laws. *SIAM/ASA Journal on Uncertainty Quantification, 9 (2011), pp. 615-646*, 2015.

[8] Jean Dolbeault, Clément Mouhot, and Christian Schmeiser. Hypocoercivity for linear kinetic equations conserving mass. *Transactions of the American Mathematical Society*, 367(6):3807–3828, 2015.

[9] Renjun Duan, Massimo Fornasier, and Giuseppe Toscani. A kinetic flocking model with diffusion. *Communications in Mathematical Physics*, 300(1):95–145, 2010.

[10] Roger. G. Ghanem and Pol. D. Spanos. *Stochastic Finite Elements: A Spectral Approach.* Springer-Verlag, New York, 1991.

[11] David Gottlieb and Dongbin Xiu. Galerkin method for wave equations with uncertain coefficients. *Commun. Comput. Phys.*, 3(2):505–518, 2008.

[12] Thierry Goudon, Juanjo Nieto, Frédéric Poupaud, and Juan Soler. Multidimensional high-field limit of the electrostatic vlasov–poisson–fokker–planck system. *Journal of Differential Equations*, 213(2):418–442, 2005.

[13] Max D. Gunzburger, Clayton G. Webster, and Guannan Zhang. Stochastic finite element methods for partial differential equations with random input data. *Acta Numer.*, 23:521–650, 2014.

[14] Jingwei Hu and Shi Jin. A stochastic galerkin method for the boltzmann equation with uncertainty. *Journal of Computational Physics*, 315:150–168, 2016.

[15] Hyung Ju Hwang and Juhi Jang. On the vlasov-poisson-fokker-planck equation near maxwellian. *Discrete & Continuous Dynamical Systems-Series B*, 18(3), 2013.

[16] Shi. Jin. Efficient asymptotic-preserving (AP) schemes for some multiscale kinetic equations. *SIAM J. Sci. Comput.*, 21:441–454, 1999.

[17] Shi. Jin. Asymptotic preserving (AP) schemes for multiscale kinetic and hyperbolic equations: a review. *Riv. Mat. Univ. Parma*, 3:177–216, 2012.

[18] Shi Jin, Jian-Guo Liu, and Zheng Ma. Uniform spectral convergence of the stochastic galerkin method for the linear transport equations with random inputs in diffusive regime and a micro-macro decomposition based asymptotic preserving method. *Research in Math. Sci.*

[19] Shi Jin and Liu Liu. An asymptotic-preserving stochastic galerkin method for the semi-conductor boltzmann equation with random inputs and diffusive scalings. *J. Comp. Phys.*, 15:157–183, 2017.

[20] Shi Jin and Hanqing Lu. An asymptotic-preserving stochastic galerkin method for the radiative heat transfer equations with random inputs and diffusive scalings. *Journal of Computational Physics*, 334:182–206, 2017.

[21] Shi Jin, Dongbin Xiu, and Xueyu Zhu. Asymptotic-preserving methods for hyperbolic and transport equations with random inputs and diffusive scalings. *Journal of Computational Physics*, 289:35–52, 2015.

[22] Qin Li and Li Wang. Uniform regularity for linear kinetic equations with random input based on hypocoercivity. *arXiv preprint arXiv:1612.01219*, 2016.

[23] Juan Nieto, Frédéric Poupaud, and Juan Soler. High-field limit for the vlasov-poisson-fokker-planck system. *Archive for Rational Mechanics and Analysis*, 158(1):29–59, 2001.

[24] Frédéric. Poupaud and Juan. Soler. Parabolic limit and stability of the Vlasov-Fokker-Planck system. *Math. Models Methods Appl. Sci.*, 10(7):1027–1045, 2000.

[25] Juan Soler. Asymptotic behaviour for the vlasov-poisson-foker-planck system. *Nonlinear Analysis: Theory, Methods & Applications*, 30(8):5217–5228, 1997.

[26] Tao Tang and Tao Zhou. Convergence analysis for stochastic collocation methods to scalar hyperbolic equations with a random wave speed. *Commun. Comput. Phys.*, 8(1):226–248, 2010.

[27] Cédric Villani. *Hypocoercivity.* Number 949-951. American Mathematical Soc., 2009.

[28] Dongbin. Xiu. *Numerical Methods for Stochastic Computations.* Princeton University Press, New Jersey, 2010.

[29] Dongbin Xiu and George Em Karniadakis. The wiener–askey polynomial chaos for stochastic differential equations. *SIAM Journal on Scientific Computing*, 24(2):619–644, 2002.

[30] Tao Zhou and Tao Tang. Convergence analysis for spectral approximation to a scalar transport equation with a random wave speed. *J. Comput. Math.*, 30(6):643–656, 2012.

[31] Yuhua Zhu and Shi Jin. The vlasov-poisson-fokker-planck system with uncertainty and a one-dimensional asymptotic preserving method. *Preprint*, 2016.